# Spectra properties of graphs and their quantum walks

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Continuous-time *random* walk on a graph $G$:

$$\frac{d}{dt}p_j(t) = -\sum_{k \in V} L_{j,k} p_k(t),$$

where $L$ is the Laplacian matrix of $G$.

Continuous-time *quantum* walk on a graph $G$:

$$\mathtt{i}\frac{d}{dt}\langle\psi(t)| = H\langle\psi(t)|,$$

for some Hermitian matrix $H$ associated with $G$.

**Definition (Farhi and Gutmann, 1998)**

*The continuous-time quantum walk in G is given by the transition operator*

$$U(t) := e^{-\mathrm{i}tA}$$

*where A is the adjacency matrix of G.*

Childs et. al (2002):*Exponential algorithmic speedup by quantum walk*
Constructed an oracular problem that can be solved exponentially faster on a quantum computer than on a classical computer.

Childs (2008): *Universal computation by quantum walk*

Construction of quantum gates by scattering processes.

Phenomena of the quantum walk:

- periodic at a vertex
- perfect state transfer between two vertices
- fractional revival from between two vertices
  (C., Coutinho, Tamon, Vinet, Zhan 2020)
- instantaneous uniform mixing

Tools from algebraic graph theory:

- spectral decomposition
- Cartesian product of two graphs

Result: For each of the above phenomena, and for $\epsilon > 0$,

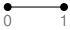
there exist graphs for which the phenomenon occurs within time $\epsilon$.

## Adjacency matrix $A$ of $G$:

$$A_{u,v} = \begin{cases} 1 & \text{if } u \text{ and } v \text{ are adjacent,} \\ 0 & \text{otherwise.} \end{cases}$$

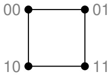
## Examples:

$P_2$ (a.k.a. 1-cube, $K_2$)



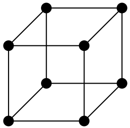

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

2-cube




$$\begin{matrix} 00 \\ 01 \\ 10 \\ 11 \end{matrix} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

3-cube



$$\begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

Example: $P_2$

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ and } A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \implies A^{2j+1} = A \text{ and } A^{2j} = \mathrm{I}, \quad \text{for } j \geq 0$$

$$\implies U(t) = e^{-\mathrm{i}tA} = \sum_{k\geq 0} \frac{(-\mathrm{i}t)^k}{k!} A^k$$

$$= \left( \sum_{j\geq 0} \frac{(-1)^j t^{2j}}{(2j)!} \right) \cdot \mathrm{I} - \mathrm{i} \left( \sum_{j\geq 0} \frac{(-1)^j t^{2j+1}}{(2j+1)!} \right) \cdot A$$

$$= \begin{bmatrix} \cos t & -\mathrm{i}\sin t \\ -\mathrm{i}\sin t & \cos t \end{bmatrix}$$

For any graph $G$,

$$U(t) = e^{-\mathrm{i}tA} = \left( \sum_{j \geq 0} \frac{(-1)^j t^{2j}}{(2j)!} A^{2j} \right) - \mathrm{i} \left( \sum_{j \geq 0} \frac{(-1)^j t^{2j+1}}{(2j+1)!} A^{2j+1} \right)$$

Observe:

- $\overline{e^{-\mathrm{i}tA}}^T = e^{\mathrm{i}t\overline{A}^T} = e^{\mathrm{i}tA}$
- $\overline{e^{-\mathrm{i}tA}}^T e^{-\mathrm{i}tA} = e^{\mathrm{i}tA} e^{-\mathrm{i}tA} = e^{-\mathrm{i}tA+\mathrm{i}tA} = \mathrm{I}$ $U(t)$ is unitary

In the continuous-time quantum walk on $G$:

State: a unit vector in $\mathbb{C}^n$.

Initial state: $e_a = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, the characteristic vector of some vertex $a$.

The state at time $t$: $U(t)e_a$.

$U(t)$ is unitary: $\left(\overline{U(t)}^T U(t)\right)_{a,a} = 1 \implies \sum_x |(U(t)e_a)_x|^2 = 1.$

The probability of $a \to x$ at time $t$ is $|U(t)_{x,a}|^2$.

### Definition

*We say the quantum walk is periodic at $a$ at time $\tau$ if $U(\tau)e_a = \alpha e_a$, for some $|\alpha| = 1$.*

$$U(\tau) = \begin{bmatrix} \alpha & 0 & 0 & \cdots & 0 \\ 0 & * & * & \cdots & * \\ 0 & * & * & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & * & * & \cdots & * \end{bmatrix}, \qquad \text{for some } |\alpha| = 1.$$

$P_2$:


$$U(\pi) = \begin{bmatrix} \cos\pi & -\mathrm{i}\sin\pi \\ -\mathrm{i}\sin\pi & \cos\pi \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$\Longrightarrow P_2$ is periodic at 0 and 1 at time $\pi$.

## Definition

*We say the quantum walk has fractional revival from a to b at time $\tau$ if*

$$U(\tau)e_a = \alpha e_a + \beta e_b,$$

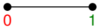
*for some $|\alpha|^2 + |\beta|^2 = 1$.*

$$U(\tau) = \begin{bmatrix} \alpha & \beta & 0 & \cdots & 0 \\ \beta & * & * & \cdots & * \\ 0 & * & * & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & * & * & \cdots & * \end{bmatrix}$$

$U(\tau)$ is symmetric

$U(\tau)$ is unitary

$P_2$:




$$U(\tfrac{\pi}{3}) = \begin{bmatrix} \cos\frac{\pi}{3} & -\mathtt{i}\sin\frac{\pi}{3} \\ -\mathtt{i}\sin\frac{\pi}{3} & \cos\frac{\pi}{3} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2}\mathtt{i} \\ -\frac{\sqrt{3}}{2}\mathtt{i} & \frac{1}{2} \end{bmatrix}$$

$\Longrightarrow$ $P_2$ has fractional revival between 0 and 1 at time $\frac{\pi}{3}$.

Special case:

$$U(\tfrac{\pi}{2}) = \begin{bmatrix} \cos\frac{\pi}{2} & -\mathtt{i}\sin\frac{\pi}{2} \\ -\mathtt{i}\sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{bmatrix} = \begin{bmatrix} 0 & -\mathtt{i} \\ -\mathtt{i} & 0 \end{bmatrix}$$

We say $P_2$ has perfect state transfer between 0 and 1 at time $\frac{\pi}{2}$.

### Definition

*We say the quantum walk has perfect state transfer from a to b at time* $\tau$ *if*

$$U(\tau)e_a = \beta e_b,$$

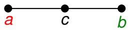
*for some* $|\beta| = 1$.

$$U(\tau) = \begin{bmatrix} 0 & \beta & 0 & \cdots & 0 \\ \beta & 0 & 0 & \cdots & 0 \\ 0 & 0 & * & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & * & \cdots & * \end{bmatrix}, \qquad \text{for some } |\beta| = 1.$$

Observe: $$U(2\tau) = U(\tau)^2 = \begin{bmatrix} \beta^2 & 0 & 0 & \cdots & 0 \\ 0 & \beta^2 & 0 & \cdots & 0 \\ 0 & 0 & * & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & * & \cdots & * \end{bmatrix}$$

Perfect state transfer at time $\tau \implies$ periodicity at time $2\tau$

**Example:** $P_3$




$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$ eigenvalues $\sqrt{2}$, 0, $-\sqrt{2}$

**(Tool: Spectral decomposition)**

$$A = \sum_{\theta} \theta E_{\theta},$$ *where $E_\theta$ is the eigenprojection onto the $\theta$-eigenspace of A.*

$P_3$:

$$A = \sqrt{2}\begin{bmatrix} 1/4 & 1/4 & \sqrt{2}/4 \\ 1/4 & 1/4 & \sqrt{2}/4 \\ \sqrt{2}/4 & \sqrt{2}/4 & 1/2 \end{bmatrix} + (0)\begin{bmatrix} 1/2 & -1/2 & 0 \\ -1/2 & 1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix} + (-\sqrt{2})\begin{bmatrix} 1/4 & 1/4 & -\sqrt{2}/4 \\ 1/4 & 1/4 & -\sqrt{2}/4 \\ -\sqrt{2}/4 & -\sqrt{2}/4 & 1/2 \end{bmatrix}$$

(Properties of $E_\theta$'s)

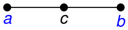
1. $E_\theta E_\eta = \begin{cases} E_\theta & \text{if } \theta = \eta, \\ 0 & \text{otherwise.} \end{cases}$
2. $\sum_\theta E_\theta = I$

$$A = \sum_\theta \theta E_\theta$$

$$\implies \quad A^k = \sum_\theta \theta^k E_\theta, \qquad \forall k \geq 0$$

$$\implies \quad U(t) = e^{-\mathrm{i}tA} = \sum_\theta e^{-\mathrm{i}t\theta} E_\theta$$

Does $P_3$ have fractional revival between $a$ and $b$?




$$U(t) = e^{-\sqrt{2}t\mathrm{i}} \begin{bmatrix} 1/4 & 1/4 & \sqrt{2}/4 \\ 1/4 & 1/4 & \sqrt{2}/4 \\ \sqrt{2}/4 & \sqrt{2}/4 & 1/2 \end{bmatrix} + e^{0} \begin{bmatrix} 1/2 & -1/2 & 0 \\ -1/2 & 1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix} + e^{\sqrt{2}t\mathrm{i}} \begin{bmatrix} 1/4 & 1/4 & -\sqrt{2}/4 \\ 1/4 & 1/4 & -\sqrt{2}/4 \\ -\sqrt{2}/4 & -\sqrt{2}/4 & 1/2 \end{bmatrix}$$

Target: $$U(\tau) = \begin{bmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{bmatrix}$$

Step 1. Group $E_\theta$'s to get desired block diagonal form:

$$E_{\sqrt{2}} + E_{-\sqrt{2}} = \begin{bmatrix} 1/4 & 1/4 & \sqrt{2}/4 \\ 1/4 & 1/4 & \sqrt{2}/4 \\ \sqrt{2}/4 & \sqrt{2}/4 & 1/2 \end{bmatrix} + \begin{bmatrix} 1/4 & 1/4 & -\sqrt{2}/4 \\ 1/4 & 1/4 & -\sqrt{2}/4 \\ -\sqrt{2}/4 & -\sqrt{2}/4 & 1/2 \end{bmatrix}, \quad E_0 = \begin{bmatrix} 1/2 & -1/2 & 0 \\ -1/2 & 1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Step 2. Find a time $\tau > 0$ such that $e^{-\sqrt{2}\tau\mathrm{i}} = e^{\sqrt{2}\tau\mathrm{i}}$.

Answer: Yes, at $\tau = \frac{\pi}{\sqrt{2}}$, $e^{-\sqrt{2}\tau\mathrm{i}} = e^{\sqrt{2}\tau\mathrm{i}} = -1$ and $U(\frac{\pi}{\sqrt{2}}) = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

Complete graph $K_n$: 

spectral decomposition:

$$A = (n-1)\begin{bmatrix} 1/n & 1/n & \cdots & 1/n \\ 1/n & 1/n & \cdots & 1/n \\ \vdots & \vdots & \ddots & \vdots \\ 1/n & 1/n & \cdots & 1/n \end{bmatrix} - 1 \begin{bmatrix} (n-1)/n & -1/n & \cdots & -1/n \\ -1/n & (n-1)/n & \cdots & -1/n \\ \vdots & \vdots & \ddots & \vdots \\ -1/n & -1/n & \cdots & (n-1)/n \end{bmatrix}$$

Fail Step 1 $\Longrightarrow$ $K_n$ has no fractional revival for $n \geq 3$.

Paths $P_n$:

Step 1 for $P_5$:

$$U(t) = \frac{e^{t\mathrm{i}}}{4}\begin{bmatrix} 1 & -1 & -1 & 1 & 0 \\ -1 & 1 & 1 & -1 & 0 \\ -1 & 1 & 1 & -1 & 0 \\ 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \frac{e^{-t\mathrm{i}}}{4}\begin{bmatrix} 1 & -1 & 1 & -1 & 0 \\ -1 & 1 & -1 & 1 & 0 \\ 1 & -1 & 1 & -1 & 0 \\ -1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} +$$

$$\frac{e^{-\sqrt{3}t\mathrm{i}}}{12}\begin{bmatrix} 1 & 1 & \sqrt{3} & \sqrt{3} & 2 \\ 1 & 1 & \sqrt{3} & \sqrt{3} & 2 \\ \sqrt{3} & \sqrt{3} & 3 & 3 & 2\sqrt{3} \\ \sqrt{3} & \sqrt{3} & 3 & 3 & 2\sqrt{3} \\ 2 & 2 & 2\sqrt{3} & 2\sqrt{3} & 4 \end{bmatrix} + \frac{e^{\sqrt{3}t\mathrm{i}}}{12}\begin{bmatrix} 1 & 1 & -\sqrt{3} & -\sqrt{3} & 2 \\ 1 & 1 & -\sqrt{3} & -\sqrt{3} & 2 \\ -\sqrt{3} & -\sqrt{3} & 3 & 3 & -2\sqrt{3} \\ -\sqrt{3} & -\sqrt{3} & 3 & 3 & -2\sqrt{3} \\ 2 & 2 & -2\sqrt{3} & -2\sqrt{3} & 4 \end{bmatrix} +$$

$$\frac{e^{0}}{12}\begin{bmatrix} 4 & 4 & 0 & 0 & -4 \\ 4 & 4 & 0 & 0 & -4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -4 & -4 & 0 & 0 & 4 \end{bmatrix}$$

Step 2: Find $\tau > 0$ such that $e^{\tau\mathrm{i}} = e^{-\tau\mathrm{i}}$ and $e^{0} = e^{\sqrt{3}\tau\mathrm{i}} = e^{-\sqrt{3}\tau\mathrm{i}}$. Fail!

**Theorem (C., Coutinho, Tamon, Vinet, Zhan 2019)**

*The path $P_n$ has fractional revival if and only if $n = 2, 3, 4$.*

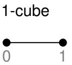
## (Tool: Cartesian product $G \,\square\, H$)

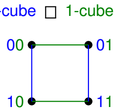
*The cartesian product of graphs $G$ and $H$ has*

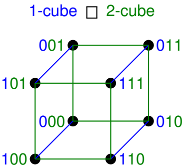
- *vertex set:* $V(G) \times V(H)$
- *edges:* $(u_1, v_1) \sim (u_2, v_2)$ *if* $u_1 \sim u_2$ *and* $v_1 = v_2$, *or* $u_1 = u_2$ *and* $v_1 \sim v_2$.

*The adjacency matrix of $G \,\square\, H$ is*

$$A_{G \,\square\, H} = A_G \otimes I_{|V(H)|} + I_{|V(G)|} \otimes A_H.$$




$$\begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

Recursive structure: 1-cube $\square$ $(n-1)$-cube gives $n$-cube, for $n \geq 2$.

Transition matrix of $G \,\square\, H$:

$$\begin{aligned} U_{G \,\square\, H}(t) &= e^{-\mathrm{i}t(A_G \otimes \mathrm{I} + \mathrm{I} \otimes A_H)} \\ &= \left(e^{-\mathrm{i}tA_G} \otimes \mathrm{I}\right)\left(\mathrm{I} \otimes e^{-\mathrm{i}tA_H}\right) \\ &= e^{-\mathrm{i}tA_G} \otimes e^{-\mathrm{i}tA_H} \\ &= U_G(t) \otimes U_H(t) \end{aligned}$$

For $n$-cubes:

$$U(t) = \begin{bmatrix} \cos t & -\mathrm{i}\sin t \\ -\mathrm{i}\sin t & \cos t \end{bmatrix}^{\otimes n}$$

Perfect state transfer: $$U(\tfrac{\pi}{2}) = \begin{bmatrix} 0 & -\mathrm{i} \\ -\mathrm{i} & 0 \end{bmatrix}^{\otimes n}$$

Periodicity: $$U(\pi) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}^{\otimes n}$$

Definition

*The continuous-time quantum walk in G is instantaneous uniform mixing at time $\tau$ if*

$$|U(\tau)_{u,v}|^2 = \frac{1}{|V(G)|}, \qquad \forall u, v.$$

*We say $U(\tau)$ is flat.*

1-cube:

$$U(t) = \begin{bmatrix} \cos t & -\mathrm{i}\sin t \\ -\mathrm{i}\sin t & \cos t \end{bmatrix}$$

At time $\tau = \frac{\pi}{4}$

$$U(\frac{\pi}{4}) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -\mathrm{i} \\ -\mathrm{i} & 1 \end{bmatrix}$$

Observe: If $G$ admits instantaneous uniform mixing then $\langle A \rangle$ contains a flat unitary matrix.

Complete graph $K_n$:

$$U(t) = e^{-\mathrm{i}(n-1)t} \begin{bmatrix} 1/n & 1/n & \cdots & 1/n \\ 1/n & 1/n & \cdots & 1/n \\ \vdots & \vdots & \ddots & \vdots \\ 1/n & 1/n & \cdots & 1/n \end{bmatrix} + e^{\mathrm{i}t} \begin{bmatrix} (n-1)/n & -1/n & \cdots & -1/n \\ -1/n & (n-1)/n & \cdots & -1/n \\ \vdots & \vdots & \ddots & \vdots \\ -1/n & -1/n & \cdots & (n-1)/n \end{bmatrix}$$

$$\frac{1}{\sqrt{n}} = |U(t)_{1,2}| = \frac{|e^{-\mathrm{i}(n-1)t} - e^{\mathrm{i}t}|}{n} \leq \frac{2}{n} \qquad \Longrightarrow \qquad n \leq 4$$

Conclusion: instantaneous uniform mixing does not occur in $K_n$, for $n \geq 5$.

*n*-cube:

$$U_{n\text{-cube}}(\frac{\pi}{4}) = \frac{1}{\sqrt{2}^n}\begin{bmatrix} 1 & -\mathtt{i} \\ -\mathtt{i} & 1 \end{bmatrix}^{\otimes n}$$

Conclusion: For $n \geq 1$,

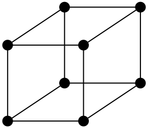
- *n*-cube has instantaneous uniform mixing at time $\frac{\pi}{4}$.
- *n*-cube has perfect state transfer between antipodal vertices at time $\frac{\pi}{2}$.
- *n*-cube is periodic at every vertex at time $\pi$.

Nice properties of *n*-cube:

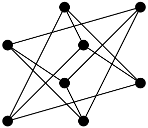
- recursive construction
- vertex transitive
- distance regular

## Distance graphs of 3-cube:

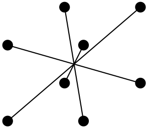
$G_1 = $ 3-cube



$G_2$



$G_3$



$$A_1 = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix} \quad A_2 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \quad A_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Distance-regular graph: $\langle A \rangle = \mathrm{span}\{\mathrm{I}, A_1, A_2, A_3\}$

When $\langle A \rangle = \mathrm{span}\{\mathrm{I}, A_1, A_2, \ldots, A_d\}$:

- Each $A_j$ is a polynomial in $A$, they form a commuting set of symmetric matrices.
- $A_0 = \mathrm{I}, A_1, \ldots, A_d$ are simultaneously diagonalizable.
- Let $E_s$ be the projection matrix onto their $s$-th common eigenspace and

$$A_r E_s = p_r(s) E_s, \qquad \text{for } r, s = 0, \ldots, d.$$

That is, $p_r(0), p_r(1), \ldots, p_r(d)$ are eigenvalues of $A_r$.

Matrix of eigenvalues:

$$\begin{array}{c} \\ E_0 \\ E_1 \\ \\ E_d \end{array} \begin{array}{c} \begin{array}{cccc} \mathrm{I} & A_1 & & A_d \end{array} \\ \begin{bmatrix} 1 & p_1(0) & \ldots & p_d(0) \\ 1 & p_1(1) & \ldots & p_d(1) \\ \vdots & \vdots & \ddots & \vdots \\ 1 & p_1(d) & \ldots & p_d(d) \end{bmatrix} \end{array}$$

$n$-cube $\langle A_{n\text{-cube}} \rangle = \mathrm{span}\{\mathrm{I}, A_1, \ldots, A_n\}$:

- Any graph in $\langle A_{n\text{-cube}} \rangle$ shares the same set of projection matrices $E_0, E_1, \ldots, E_n$.
- Eigenvalues:

$$p_r(s) = \sum_{h=0}^{r} (-2)^h \binom{n-h}{r-h} \binom{s}{h}, \qquad s = 0, \ldots, n.$$

  Krawtchouk polynomials $p_r(x)$.

- $n$-cube admits instantaneous uniform mixing $\Longrightarrow$ $\langle A_{n\text{-cube}} \rangle$ contains a flat unitary matrix

Question: What about other graphs in $\langle A_{n\text{-cube}} \rangle$?

Idea: Cheat! Find a 01-matrix $M$ in $\langle A_{n\text{-cube}} \rangle$ satisfying

$$e^{-\mathrm{i}\tau M} = \frac{e^{i\beta}}{\sqrt{2}^n} \begin{bmatrix} 1 & \pm\mathrm{i} \\ \pm\mathrm{i} & 1 \end{bmatrix}^{\otimes n}, \quad \text{for some } \tau \text{ and } \beta.$$

Eigenvalues: $\frac{e^{i\beta}}{\sqrt{2}^n} \begin{bmatrix} 1 & \pm\mathrm{i} \\ \pm\mathrm{i} & 1 \end{bmatrix}^{\otimes n} E_s = e^{\mathrm{i}\beta} e^{\pm\mathrm{i}\pi\frac{n-2s}{4}} E_s, \quad \text{for } s = 0, 1, \ldots, n.$

Let $\theta_0, \theta_1, \ldots, \theta_n$ be the eigenvalues of $M$. Then

$$
\begin{aligned}
& e^{-i\tau M} && = e^{i\beta} \frac{1}{\sqrt{2}^n} \begin{pmatrix} 1 & \pm\mathrm{i} \\ \pm\mathrm{i} & 1 \end{pmatrix}^{\otimes n} \\
\iff \quad & e^{-i\tau\theta_s} && = e^{i\beta} e^{\pm\mathrm{i}\pi(n-2s)/4}, \qquad \forall s \\
\iff \quad & \tau\theta_s && = \pm\pi(n-2s)/4 + \beta \pmod{2\pi}, \qquad \forall s \\
\iff \quad & \tau(\theta_s - \theta_{s-1}) && = \pm\frac{\pi}{2} \pmod{2\pi}, \qquad \forall s > 0
\end{aligned}
$$

Example: 4-cube and $\tau = \frac{\pi}{4}$

$$\tau(\theta_s - \theta_{s-1}) = \pm\tfrac{\pi}{2} \pmod{2\pi} \iff (\theta_s - \theta_{s-1}) = \pm 2 \pmod 8$$

$$\begin{array}{c} \\ \theta_0 \\ \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{array} \begin{array}{c} \begin{array}{ccccc} \mathrm{I} & A_1 & A_2 & A_3 & A_4 \end{array} \\ \begin{bmatrix} 1 & 4 & 6 & 4 & 1 \\ 1 & 2 & 0 & -2 & -1 \\ 1 & 0 & -2 & 0 & 1 \\ 1 & -2 & 0 & 2 & -1 \\ 1 & -4 & 6 & -4 & 1 \end{bmatrix} \end{array}$$

Both $G_1$ and $G_3$ admit instantaneous uniform mixing at $\frac{\pi}{4}$.

**Theorem**

*For even $n$ and odd $r$, $G_r$ admits instantaneous uniform mixing at time $\frac{\pi}{4}$.*

Faster $\tau = \frac{\pi}{8}$: want $\theta_s - \theta_{s-1} = \pm 2^2 \pmod{2^4}$

24-cube: $\theta_s - \theta_{s-1} \pmod{16}$

$E_s$

| | | | | | | | | | | | | | | | $A_{15}$ | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 8 | 4 | 8 | 2 | 8 | 4 | 8 | 15 | 0 | 8 | 0 | 12 | 0 | 8 | 0 | 15 | 8 | 4 | 8 | 2 | 8 | 4 | 8 | 1 |
| 1 | 6 | 6 | 14 | 12 | 10 | 2 | 2 | 5 | 12 | 12 | 12 | 0 | 4 | 4 | 4 | 11 | 14 | 14 | 6 | 4 | 2 | 10 | 10 | 15 |
| 1 | 4 | 12 | 12 | 2 | 12 | 12 | 4 | 15 | 8 | 8 | 8 | 12 | 8 | 8 | 8 | 15 | 4 | 12 | 12 | 2 | 12 | 12 | 4 | 1 |
| 1 | 2 | 6 | 10 | 12 | 14 | 2 | 6 | 5 | 4 | 12 | 4 | 0 | 12 | 4 | 12 | 11 | 10 | 14 | 2 | 4 | 6 | 10 | 14 | 15 |
| 1 | 0 | 4 | 0 | 2 | 0 | 4 | 0 | 15 | 0 | 8 | 0 | 12 | 0 | 8 | 0 | 15 | 0 | 4 | 0 | 2 | 0 | 4 | 0 | 1 |
| 1 | 14 | 6 | 6 | 12 | 2 | 2 | 10 | 5 | 12 | 12 | 12 | 0 | 4 | 4 | 4 | 11 | 6 | 14 | 14 | 4 | 10 | 10 | 2 | 15 |
| 1 | 12 | 12 | 4 | 2 | 4 | 12 | 12 | 15 | 8 | 8 | 8 | 12 | 8 | 8 | 8 | 15 | 12 | 12 | 4 | 2 | 4 | 12 | 12 | 1 |
| 1 | 10 | 6 | 2 | 12 | 6 | 2 | 14 | 5 | 4 | 12 | 4 | 0 | 12 | 4 | 12 | 11 | 2 | 14 | 10 | 4 | 14 | 10 | 6 | 15 |
| 1 | 8 | 4 | 8 | 2 | 8 | 4 | 8 | 15 | 0 | 8 | 0 | 12 | 0 | 8 | 0 | 15 | 8 | 4 | 8 | 2 | 8 | 4 | 8 | 1 |
| 1 | 6 | 6 | 14 | 12 | 10 | 2 | 2 | 5 | 12 | 12 | 12 | 0 | 4 | 4 | 4 | 11 | 14 | 14 | 6 | 4 | 2 | 10 | 10 | 15 |
| 1 | 4 | 12 | 12 | 2 | 12 | 12 | 4 | 15 | 8 | 8 | 8 | 12 | 8 | 8 | 8 | 15 | 4 | 12 | 12 | 2 | 12 | 12 | 4 | 1 |
| 1 | 2 | 6 | 10 | 12 | 14 | 2 | 6 | 5 | 4 | 12 | 4 | 0 | 12 | 4 | 12 | 11 | 10 | 14 | 2 | 4 | 6 | 10 | 14 | 15 |
| 1 | 0 | 4 | 0 | 2 | 0 | 4 | 0 | 15 | 0 | 8 | 0 | 12 | 0 | 8 | 0 | 15 | 0 | 4 | 0 | 2 | 0 | 4 | 0 | 1 |
| 1 | 14 | 6 | 6 | 12 | 2 | 2 | 10 | 5 | 12 | 12 | 12 | 0 | 4 | 4 | 4 | 11 | 6 | 14 | 14 | 4 | 10 | 10 | 2 | 15 |
| 1 | 12 | 12 | 4 | 2 | 4 | 12 | 12 | 15 | 8 | 8 | 8 | 12 | 8 | 8 | 8 | 15 | 12 | 12 | 4 | 2 | 4 | 12 | 12 | 1 |
| 1 | 10 | 6 | 2 | 12 | 6 | 2 | 14 | 5 | 4 | 12 | 4 | 0 | 12 | 4 | 12 | 11 | 2 | 14 | 10 | 4 | 14 | 10 | 6 | 15 |
| 1 | 8 | 4 | 8 | 2 | 8 | 4 | 8 | 15 | 0 | 8 | 0 | 12 | 0 | 8 | 0 | 15 | 8 | 4 | 8 | 2 | 8 | 4 | 8 | 1 |
| 1 | 6 | 6 | 14 | 12 | 10 | 2 | 2 | 5 | 12 | 12 | 12 | 0 | 4 | 4 | 4 | 11 | 14 | 14 | 6 | 4 | 2 | 10 | 10 | 15 |
| 1 | 4 | 12 | 12 | 2 | 12 | 12 | 4 | 15 | 8 | 8 | 8 | 12 | 8 | 8 | 8 | 15 | 4 | 12 | 12 | 2 | 12 | 12 | 4 | 1 |
| 1 | 2 | 6 | 10 | 12 | 14 | 2 | 6 | 5 | 4 | 12 | 4 | 0 | 12 | 4 | 12 | 11 | 10 | 14 | 2 | 4 | 6 | 10 | 14 | 15 |
| 1 | 0 | 4 | 0 | 2 | 0 | 4 | 0 | 15 | 0 | 8 | 0 | 12 | 0 | 8 | 0 | 15 | 0 | 4 | 0 | 2 | 0 | 4 | 0 | 1 |
| 1 | 14 | 6 | 6 | 12 | 2 | 2 | 10 | 5 | 12 | 12 | 12 | 0 | 4 | 4 | 4 | 11 | 6 | 14 | 14 | 4 | 10 | 10 | 2 | 15 |
| 1 | 12 | 12 | 4 | 2 | 4 | 12 | 12 | 15 | 8 | 8 | 8 | 12 | 8 | 8 | 8 | 15 | 12 | 12 | 4 | 2 | 4 | 12 | 12 | 1 |
| 1 | 10 | 6 | 2 | 12 | 6 | 2 | 14 | 5 | 4 | 12 | 4 | 0 | 12 | 4 | 12 | 11 | 2 | 14 | 10 | 4 | 14 | 10 | 6 | 15 |

| Nutrition Facts/Valeur nutritive | |
|---|---|
| per $n$-cube | |
| **Calories/Calories** | $\pm$? |
| ⋮ | ⋮ |
| **Carbohydrate/Glucides** | 5g |
| Fibre/Fibres | 0g |
| Sugars | 5g |
| ⋮ | ⋮ |

INGREDIENTS: continuous-time quantum walk, $p_r(s)$ (Krawtchouk polynomials), spectral decomposition, *number theory (Lucas' Theorem, Kummer's Theorem)*, goos paper, snacks, teabags.

**Theorem**

*For $k \geq 2$, let $n = 2^{k+2} - 8$. The graph $G_{2^{k+1}-1}$ admits instantaneous uniform mixing at $\pi/2^k$.*

Observe:

$$\left( \frac{1}{\sqrt{2}^n} \begin{bmatrix} 1 & \pm \mathrm{i} \\ \pm \mathrm{i} & 1 \end{bmatrix}^{\otimes n} \right)^2 = \begin{bmatrix} 0 & \pm \mathrm{i} \\ \pm \mathrm{i} & 0 \end{bmatrix}^{\otimes n}$$

Conclusion:

$$U(\tau) = \frac{1}{\sqrt{2}^n} \begin{bmatrix} 1 & \pm \mathrm{i} \\ \pm \mathrm{i} & 1 \end{bmatrix}^{\otimes n} \quad \text{Inst. uniform mixing}$$

$$U(2\tau) = \begin{bmatrix} 0 & \pm \mathrm{i} \\ \pm \mathrm{i} & 0 \end{bmatrix}^{\otimes n} \quad \text{perfect state transfer}$$

$$U(4\tau) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}^{\otimes n} \quad \text{periodicity}$$

**Theorem**

*For $k \geq 2$, let $n = 2^{k+2} - 8$. The graph $G_{2^{k+1}-1}$ admits perfect state transfer at $\pi/2^{k-1}$.*

Fractional revival in $\langle A_{n\text{-cube}} \rangle$:

$$\alpha A_0 + \beta A_n = \begin{bmatrix} \alpha & 0 & \cdots & \cdots & 0 & \beta \\ 0 & \alpha & \cdots & \cdots & \beta & 0 \\ 0 & 0 & \ddots & \iddots & 0 & 0 \\ 0 & 0 & \iddots & \ddots & 0 & 0 \\ 0 & \beta & \cdots & \cdots & \alpha & 0 \\ \beta & 0 & \cdots & \cdots & 0 & \alpha \end{bmatrix} \in \langle A_{n\text{-cube}} \rangle$$

**Theorem (C. Coutinho, Tamon, Vinet, Zhan 2020)**

*For $k \geq 4$, let $n = (2^{k-1} + 1)2^{k+2} + 3$ and $r = 2^{k+3} + 3$. The graph $G_r$ has fractional revival at time $\frac{\pi}{2^k}$.*

**Theorem**

*For $\epsilon > 0$, there exist graphs having* $\begin{cases} \textit{periodicity} \\ \textit{fractional revival} \\ \textit{perfect state transfer} \\ \textit{instantaneous uniform mixing} \end{cases}$ *within time $\epsilon$.*

Perfect state transfer is monogamous

**Theorem (Godsil 2011)**

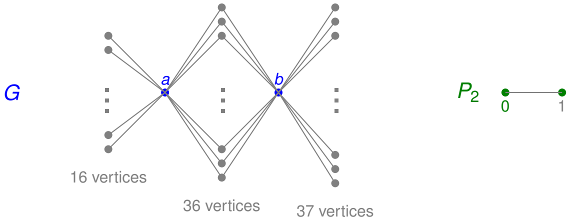
*If $G$ has perfect state transfer from $a$ to $b$ and also from $a$ to $c$ then $a = c$.*

What about fractional revival?

(Tool: Cartesian product)

*Suppose, at time $\tau$, $G_1$ has fractional revival between $a$ and $b$ and $G_2$ is periodic at vertex $u$. Then $G_1 \square G_2$ has fractional revival between $(a, u)$ and $(b, u)$ at time $\tau$.*

Example (Zhang):




| | $G$ | $P_2$ | $G \square P_2$ |
|---|---|---|---|
| $\tau = \pi$ | FR between $a$ and $b$ | periodic at 0 | FR between $(a, 0)$ and $(b, 0)$ |
| $\tau = \frac{2\pi}{5}$ | periodic at $a$ | FR between 0 and 1 | FR between $(a, 0)$ and $(a, 1)$ |

Conclusion: Fractional revival is polygamous!


*THANK YOU*!

*(Photo and drawings are obtained from Creative Commons.)*