UNIVERSITY OF MANITOBA

April 16th, 2015

Final Exam

NO. OF PAGES: 7

LOCATION: 402 Allen Bldg.

DEPT. AND COURSE NO.: PHYS 2380

TIME: 13:30 to 16:30 (3 hours)

EXAMINATION: Quantum Physics 1

Examiners: K.S. Sharma

Instructions:

- Answer all questions.
- Start each question on a new page.
- Show all of your steps in arriving at the solution. Do not just state the answer. State your arguments clearly and indicate any assumptions you make.
- Include a sentence or two at the end of each problem summarizing your solution.
- Each part of a question carries equal weight.

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1. A cavity is maintained at a temperature of 1650 *K*.
   a) At what wavelength is the peak of the radiated energy?
   b) What is the ratio of the radiance at twice the wavelength determined in part (a) to the radiance at the wavelength where the radiance is a maximum?
   c) At what rate does energy escape from a hole in the walls of the cavity with a diameter of 1.00 *mm*?

   (2+4+4 marks)

2. The de Broglie wavelength of a particle is defined to be: $\lambda = h/p$.
   a) Show that for all energies large and small that :

   $$\lambda = \frac{hc}{E_k\left(1 + 2mc^2/E_k\right)^{1/2}}.$$

   b) Show that in the extreme relativistic limit that this expression approaches that for a photon of similar energy.
   c) Show that in the classical limit that this expression reduces to $\lambda = h/(mv)$ as expected.

   (3+4+4 Marks)

3. The wave function for the harmonic oscillator potential $\left(V(x) = \frac{m\omega^2}{2}x^2\right)$ for n=1 is given by:

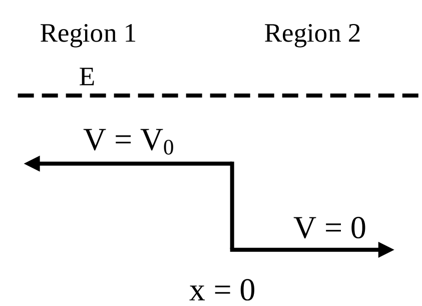
   Where C is a constant and x is the position of the particle.

   $$\Psi_1(x,t) = A_1 u e^{-u^2/2} e^{-iEt/\hbar}$$
   $$where\ u = \left(\sqrt{m\omega/\hbar}\right)x \text{ and}$$
   $$E = \tfrac{3}{2}\hbar\omega$$

   a) Show that $A_1 = (4m\omega/\pi\hbar)^{1/4}$ by normalizing the wave function.
   b) Without computing the integrals, argue from the form of the integrand that $\langle x \rangle$ and $\langle p \rangle$ are zero.
   c) Show that the average value for position-squared: $\langle x^2 \rangle$ for this state is $\frac{3}{2}\left(\frac{\hbar}{m\omega}\right)$.
   d) Show that the average value for momentum-squared: $\langle p^2 \rangle$ for this state is $\tfrac{3}{2}(\hbar m\omega)$
   e) Using the results from parts (c) and (d) obtain values for the average values for potential and kinetic energies and their sum. Express your answers in terms of $\hbar$ and ω.

   (4+3+4+4+3 marks)

4. A wave is incident from the left on the potential function, V(x), shown here:

   a) Write down the "space-part" wave function for a particle with energy $E > V_0$ in both regions. Identify the incident and reflected components in these wave functions. Do any of the components have a zero amplitude (explain)?
   b) What are the corresponding "time-parts" of the wave function?
   c) Given that $V_0 = \frac{5}{9}E$ provide expressions for the wave number (*k*) in each region. Express of them in terms of the other. What is the ratio of k_1 to k_2?
   d) Apply the boundary conditions and determine the amplitudes for all the component waves in terms of one of them.
   e) From the coefficients determined in part (d) derive an expression for the probability density as a function of x.
   f) Sketch the probability density and locate the maxima, minima and constant levels if any. Assign relative values to maxima, minima and constant levels.
   g) <u>Directly</u> calculate the transmission coefficient *T* using your results from part (d).
   h) <u>Directly</u> calculate the reflection coefficient *R* using your results from part (d).

   (3+3+3+3+3+3+3+3 marks)

5. The normalized wave functions for the n=2 state of the hydrogen atom are:

$$\psi_{nlm} = R_{nl}\left(r\right)\Theta_{lm}\left(\theta\right)\Theta\left(\phi\right) \; where:$$

$$R_{20} = \frac{1}{\sqrt{2a_0^3}}\left(1 - \frac{r}{2a_0}\right)e^{-r/2a_0}$$

$$R_{21} = \frac{1}{2\sqrt{6a_0^3}}\left(\frac{r}{a_0}\right)e^{-r/2a_0}$$

$$and \; the \; angular \; dependence \; is \; given \; by:$$

$$\Theta_{11}\Phi_1 = \sqrt{\frac{3}{8\pi}}\sin\theta e^{i\phi}$$

$$\Theta_{10}\Phi_0 = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$\Theta_{11}\Phi_{-1} = \sqrt{\frac{3}{8\pi}}\sin\theta e^{-i\phi}$$

where a_0 is the Bohr radius.

a) For what value of *r* will the radial probability distribution corresponding to the state 210 have its maximum value? Express your answer in units of the Bohr radius, a_0.

b) Sketch a polar diagram representing the modulation factors that result from the 3 possibilities for the angular wave functions. Comment on the shapes of these diagrams and what they tell us about the orientation of the planes of the orbits.

c) Show that R_{21} normalized as given.

d) Show that $\Theta_{11}\Phi_1$ is normalized as given.

(4+4+4+4 marks)

The End

Appendix: Some information from the text and lectures:

Special Relativity:

Relativistic momentum and energy:

$$\vec{p} = \gamma m \vec{\mathrm{v}}$$
$$E = \gamma mc^2 = mc^2 + K$$
$$E^2 = c^2 p^2 + m^2 c^4$$

Electromagnetic radiation:

Power received by a detector from a wave:

$$P = \left(\frac{1}{\mu_0 c}\right) E_0^2 A \sin^2(kz - \omega t + \phi)$$
$$P_{ave} = \frac{1}{T_0}\int_0^{T_0} P dt = \frac{E_0^2 A}{2\mu_0 c}, \quad I = \frac{P_{ave}}{A} = \frac{E_0^2}{2\mu_0 c}$$

Where P is the instantaneous power, P_{ave} is the average power delivered to a detector of area A and I is the intensity of the light.

Interference and diffraction:

Pattern Type	Bright Fringes	Dark Fringes
Single slit (width *w*)	$\frac{w}{2}\sin\theta = m\lambda$	$\frac{w}{2}\sin\theta = \left(m + \frac{1}{2}\right)\lambda$
Double slit (spacing *d*)	$d\sin\theta = m\lambda$	
Grating (lines spaced *d* apart)	$d\sin\theta = m\lambda$	
Bragg (layers of atoms *d* apart)	$2d\sin\theta = m\lambda$	
Circular object		First fringe at $1.22\lambda/d$

Photons and light:

$$\lambda\nu = c$$
$$E_{ph} = h\nu = cp_{ph}$$
$$p_{ph} = h/\lambda$$

Photoelectric effect: $K = h\nu - \phi = eV_s$

Where K is the kinetic energy of the emitted electrons, ϕ is the work function of the material and V_s is the stopping potential.

Black body radiation:

$$I = \sigma T^4$$
$$\lambda_{\max} T = 2.898 \times 10^{-3} m \bullet K$$
$$u(\lambda) = \left(\frac{8\pi hc}{\lambda^5}\right)\left[\frac{1}{e^{hc/\lambda kT} - 1}\right]$$
$$dI = \frac{c}{4} u(\lambda) d\lambda$$

Where *u(λ)* is the energy density, *h* is Planck's constant, *k* is the Botzmann constant and *c* is the speed of light.

Compton Scattering: $\lambda' - \lambda = \frac{h}{m_e c}\left(1 - \cos\theta\right)$

Bremsstrahlung: $\lambda_{\min} = \frac{hc}{eV}$

Wavelike properties of particles:

De Broglie wavelength: $\lambda = h/p$

Hiesenberg uncertainty relationships:

$$\Delta E \Delta t \geq \frac{\hbar}{2} \qquad \Delta x = \left(\langle x^2 \rangle - \langle x \rangle^2\right)^{1/2}$$
$$\Delta p_x \Delta x \geq \frac{\hbar}{2} \qquad \Delta p = \left(\langle p^2 \rangle - \langle p \rangle^2\right)^{1/2}$$

Wave packets:

$$v_{group} = \frac{d\omega}{dk}$$

$$v_{phase} = \frac{\omega}{k}$$

$$p = h/\lambda = \hbar k$$

$$\hbar\omega = \frac{p^2}{2m} = \frac{\hbar^2 k^2}{2m}$$

Quantum Mechanics:

Schrödinger equation:

Complete: $-\frac{\hbar^2}{2m}\frac{\partial^2\Psi(x,t)}{\partial x^2} + V(x,t)\Psi(x,t) = i\hbar\frac{\partial\Psi(x,t)}{\partial t}$

Time independent: $-\frac{\hbar^2}{2m}\frac{\partial^2\psi(x)}{\partial x^2} + V(x)\psi(x) = E\psi(x)$

Probability Current: $S(x,t) = \frac{i\hbar}{2m}\left\{\Psi\frac{\partial\Psi^*}{\partial x} - \Psi^*\frac{\partial\Psi}{\partial x}\right\}$

Normalization (1-D): $\int_{-\infty}^{+\infty}\Psi^*\Psi dx = 1$

Normalization (3-D): $\int_0^{+\infty}\int_0^{\pi}\int_0^{2\pi}\Psi^*\Psi\, r^2\sin\theta\, dr\, d\theta\, d\phi = 1$ or $\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\Psi^*\Psi\, dx\, dy\, dz = 1$

H-atom: $\Psi(r,\theta,\phi) = R(r)\Theta(\theta)\Phi(\phi)$ each of these components may be normalized individually.

Radial probability distribution: $P(r) = r^2 R^*(r) R(r)$

Momentum operator: $p_{op} = -i\hbar\frac{\partial}{\partial x}$

Nuclear Physics:

$$R = R_0 A^{1/3} = 1.2 A^{1/3}\, fm$$

$$B = \left[Zm\left({}_1^1H\right) + Nm_n - m\left({}_z^AX\right)\right]c^2$$

$$Q = [M_{parent} - M_{Daughter} - M_{emitted}]c^2$$

$$N = N_0 e^{-\lambda t}$$

$$A = \lambda N$$

Nuclear Atom:

$$F = \frac{(ze)(Ze)}{4\pi\varepsilon_0 r^2}$$

$$U = \frac{(ze)(Ze)}{4\pi\varepsilon_0 r}$$

Rutherford Scattering:

$$b = \frac{zZ}{2K}\frac{e^2}{4\pi\varepsilon_0}\cot\tfrac{1}{2}\theta$$

$$\tfrac{1}{2}mv^2 = \tfrac{1}{2}\left(\frac{b^2v^2}{r_{\min}^2}\right) + \frac{e^2}{4\pi\varepsilon_0}\frac{zZ}{r_{\min}}$$

$$d = \frac{1}{4\pi\varepsilon_0}\frac{zZe^2}{K}$$

Bohr model: $E_n = -\frac{me^4}{32\pi^2\varepsilon_0^2\hbar^2}\frac{1}{n^2} = -\frac{13.6Z_{eff}^2}{n^2}eV$

$$\frac{1}{\lambda} = R\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right)$$

$$R = R_\infty\left(\frac{1}{1+m/M}\right)$$

$$R_\infty = \frac{mk^2e^4}{4\pi c\hbar^3}\ where\ k = \frac{1}{4\pi\varepsilon_0}$$

Bohr radius: $a_0 = \frac{\hbar^2}{4\pi\varepsilon_0 me^2}$

$$r_n = n^2 a_0 / Z$$

X-rays:

K-series: $E_{photon} = (13.6eV)\left(\frac{1}{1^2} - \frac{1}{n^2}\right)(Z-1)^2$

L-series: $E_{photon} = (13.6eV)\left(\frac{1}{2^2} - \frac{1}{n^2}\right)(Z-3)^2$

M-series: $E_{photon} = (13.6eV)\left(\frac{1}{3^2} - \frac{1}{n^2}\right)(Z-5)^2$

Some useful mathematical relations:

$$\sqrt{\left(1-u^2/c^2\right)} \approx 1-\frac{1}{2}\frac{u^2}{c^2}$$

$$\frac{1}{\sqrt{\left(1-u^2/c^2\right)}} \approx 1+\frac{1}{2}\frac{u^2}{c^2}$$

For $u^2/c^2 << 1$

Some useful trigonometric relations:

$$\sin(A+B)=\sin(A)\cos(B)+\cos(A)\sin(B)$$

$$\cos(A+B)=\cos(A)\cos(B)-\sin(A)\sin(B)$$

$$\sin(2A)=2\sin(A)\cos(A)$$

$$\sin^2(A)=\frac{1-\cos(2A)}{2}$$

$$\cos^2(A)=\frac{1+\cos(2A)}{2}$$

$$e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$$

Some useful Integrals:

$$\int \sin^2(u)\,du=\frac{1}{2}\left(u-\sin u\cos u\right)$$

$$\int \sin u\cos u\,du=\frac{1}{2}\sin^2 u$$

$$\int \cos^2 u\,du=\frac{1}{2}\left(u+\sin u\cos u\right)$$

$$\int u\sin^2 u\,du=\frac{u^2}{4}-\frac{u\sin 2u}{4}-\frac{\cos 2u}{8}$$

$$\int u\cos^2 u\,du=\frac{u^2}{4}+\frac{u\sin 2u}{4}+\frac{\cos 2u}{8}$$

$$\int u^2\sin^2 u\,du=\frac{u^3}{6}-\left(\frac{u^2}{4}-\frac{1}{8}\right)\sin 2u-\frac{u\cos 2u}{4}$$

$$\int u^2\cos^2 u\,du=\frac{u^3}{6}+\left(\frac{u^2}{4}-\frac{1}{8}\right)\sin 2u+\frac{u\cos 2u}{4}$$

$$\int \sin^3 u\,du=-\frac{1}{3}\cos u\left[\sin^2 u+2\right]$$

$$\int \cos^3 u\,du=\frac{1}{3}\sin u\left[\cos^2 u+2\right]$$

$$\int_0^\infty u^n e^{-u}\,du=n! \quad for\ n>0$$

$$\int \cos^n u\sin u\,du=-\frac{\cos^{n+1}u}{n+1} \quad for\ n>0$$

$$\int \sin^n u\cos u\,du=\frac{\sin^{n+1}u}{n+1} \quad for\ n>0$$

Integrals forms involving $e^{-\lambda x^2}$ (note the limits of integration):

n	$I_n=\int_0^\infty x^n e^{-\lambda x^2}dx$
0	$\frac{1}{2}\pi^{1/2}\lambda^{-1/2}$
1	$\frac{1}{2}\lambda^{-1}$
2	$\frac{1}{4}\pi^{1/2}\lambda^{-3/2}$
3	$\frac{1}{2}\lambda^{-2}$

n	$I_n=\int_0^\infty x^n e^{-\lambda x^2}dx$
4	$\frac{3}{8}\pi^{1/2}\lambda^{-5/2}$
5	λ^{-3}
If n is even	$\int_{-\infty}^\infty x^n e^{-\lambda x^2}dx=2I_n$
If n is odd	$\int_{-\infty}^\infty x^n e^{-\lambda x^2}dx=0$

Constants:

Constant	Standard value	Alternate units
Speed of light	$c = 2.998\times10^{8} m/s$	
Electronic charge	$e = 1.602\times10^{-19} C$	
Boltzmann constant	$k = 1.381\times10^{-23} J/K$	$8.617\times10^{-5} eV/K$
Planck's constant	$h = 6.626\times10^{-34} J\cdot s$	$4.136\times10^{-15} eV\cdot s$
	$\hbar = 1.055\times10^{-34} J\cdot s$	$0.652\times10^{-15} eV\cdot s$
Avogadro's constant	$N_A = 6.022\times10^{23} mole^{-1}$	
Stefan-Boltzmann constant	$\sigma = 5.67\times10^{-8}\ W/m^2\cdot K^4$	
Electron mass	$m_e = 5.49\times10^{-4} u\ or\ 9.109\times10^{-31} kg$	$0.511 MeV/c^2$
Proton mass	$1.007276\ u\ or\ 1.673\times10^{-27}\ kg$	$938.3 MeV/c^2$
Neutron mass	$1.008665\ u\ or\ 1.675\times10^{-27}\ kg$	$939.6 MeV/c^2$
Mass of ^{4}He	*4.002603 u*	
Bohr radius	$a_0 = 4\pi\varepsilon_0\hbar^2 / m_e e^2 = 0.0529 nm$	
Hydrogen ionization energy	$13.6 eV$	
	$hc = 1240 ev\cdot nm$	
	$1eV = 1.602\times10^{-19} J$	
Atomic mass unit (dalton)	$1u = 931.5 MeV/c^2$	$1.661\times10^{-27} kg$
	$kT = 0.02525 eV \approx \frac{1}{40} eV$ at T=293 K	
	$\frac{1}{4\pi\varepsilon_0} = 8.988\times10^{9} N\cdot m^2\cdot C^{-2}$	
	$\frac{e^2}{4\pi\varepsilon_0} = 1.44 eV\cdot nm$	