

**April 14<sup>th</sup>, 2011****Final Exam****LOCATION: 405 Allen Bldg.****NO. OF PAGES: 6****DEPT. AND COURSE NO.: PHYS 2380****TIME: 09:00 to 12:00 (3 hours)****EXAMINATION: Quantum Mechanics 1****Examiners: K.S. Sharma****Instructions:**

- Answer all questions.
- Start each question on a new page.
- Show all of your steps in arriving at the solution. Do not just state the answer. State your arguments clearly and indicate any assumptions you make.
- Include a sentence or two at the end of each problem summarizing your solution.
- Each part of a question carries equal weight.

1. The energy density, as a function of wavelength,  $\lambda$ , and temperature,  $T$ , inside a black body

is given by  $u(\nu) = \frac{8\pi h\nu^3}{c^3} \frac{1}{e^{h\nu/kT} - 1}$  where  $h$  is Planck's constant,  $k$  is the Boltzmann constant

and  $c$  is the speed of light.

- Transform the expression for  $u(\nu)$  into an expression for the energy density as a function of wavelength,  $u(\lambda)$ ? Recall that the definition of energy density means that the power density ( $W/m^2$ ) contained in a range of frequencies  $d\nu$  is given by  $u(\nu) d\nu$  and that your results for  $u(\lambda)d\lambda$  must be consistent with this.
- The spectral radiancy of a black body is given by  $R(\nu) = \frac{c}{4}u(\nu)$ . Integrate the spectral radiancy to arrive at Stefan's law which states that the total radiancy of a black body is given by  $R_T = \sigma T^4 W / m^2$ . Provide an expression for  $\sigma$  in terms of the constants  $h$ ,  $c$  and  $k$ .

(20 marks)

2. The wave function for the harmonic oscillator potential  $\left( V(x) = \frac{C}{2}x^2 \right)$  for  $n=0$  is given by:

Where  $C$  is a constant and  $x$  is the position of the particle.

$$\Psi_0(x,t) = A_0 e^{-m\omega x^2/2\hbar} e^{-iEt/\hbar}$$

$$\text{where } \omega = \sqrt{C/m}$$

$$E = (n + 1/2)\hbar\omega$$

- Verify that this wave function is a solution to the Schrodinger Equation by directly substituting it into the equation.
- Determine  $A_0$  by normalizing the wave function.
- Evaluate the average values  $\langle x^2 \rangle$  and  $\langle p^2 \rangle$  for this state.
- Using the results from part (b) obtain values for the average values for potential and kinetic energies and their sum. Express your answers in terms of  $\hbar$  and  $\omega$ .

- e) Use the results from part (b) to calculate the uncertainty in  $x$  and  $p$  and verify that the Heisenberg uncertainty principle is satisfied.

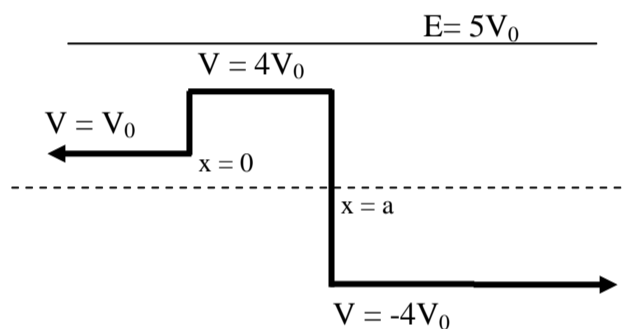
(20 marks)

3. An electron is confined in an infinite square well potential with  $V(x) = 0$  for  $0 \leq x \leq a$ , and is infinite everywhere else.

- Solve the Schrodinger equation for the wave functions of such a particle and derive an expression for the allowed values of its energy.
- Determine the amplitude necessary to normalize these wave functions.
- Derive a general expression for the probability of finding the particle (in any of these states) in the region  $x_1 \leq x \leq x_2$  where  $x_1$  and  $x_2$  lie inside the well.
- Compare your answers for part (c) for the limit at very high energies (large  $n$ ) with what you would expect classically.

(20 marks)

4. A particle is incident from the left on the potential function,  $V(x)$ , shown here:



- Write down the most general forms for the “space-part” wave function for a particle with energy  $E = 5V_0$  in region 1 ( $x < 0$ ), region 2 ( $0 < x < a$ ) and region 3 ( $x > a$ ).
- Provide expressions for the wave number ( $k$ ) in each region (you may find it helpful to express the wave numbers as multiples of the wave number in region 2).
- Identify the forward and reflected components in these wave functions. Are any of the terms obviously zero?
- What are the corresponding “time-parts” of the wave function?
- Apply the boundary conditions and determine the amplitudes for all the component waves in terms of one of them.
- From the coefficients determined in part (e) find the reflection coefficient for the incident wave (in region 1).
- From the coefficients determined in part (e) find the transmission coefficient for the incident wave (in region 1) into region 3.
- Use your results to prove that the sum of the transmission and reflection coefficients is 1.
- What is the condition for maximum transmission into region 3?

(27 marks)

**The End**

**Appendix: Some information from the text and lectures:**

**Special Relativity:**

Relativistic momentum and energy:

$$\vec{p} = \gamma m \vec{v}$$

$$E = \gamma mc^2 = mc^2 + K$$

$$E^2 = c^2 p^2 + m^2 c^4$$

**Electromagnetic radiation:**

Power received by a detector from a wave:

$$P = \left( \frac{1}{\mu_0 c} \right) E_0^2 A \sin^2(kz - \omega t + \phi)$$

$$P_{ave} = \frac{1}{T_0} \int_0^{T_0} P dt = \frac{E_0^2 A}{2\mu_0 c}, \quad I = \frac{P_{ave}}{A} = \frac{E_0^2}{2\mu_0 c}$$

Where  $P$  is the instantaneous power,  $P_{ave}$  is the average power delivered to a detector of area  $A$  and  $I$  is the intensity of the light.

**Interference and diffraction:**

Pattern Type	Bright Fringes	Dark Fringes
Single slit (width $w$ )	$\frac{w}{2} \sin \theta = m\lambda$	$\frac{w}{2} \sin \theta = (m + \frac{1}{2})\lambda$
Double slit (spacing $d$ )	$d \sin \theta = m\lambda$	
Grating (lines spaced $d$ apart)	$d \sin \theta = m\lambda$	
Bragg (layers of atoms $d$ apart)	$2d \sin \theta = m\lambda$	
Circular object		First fringe at $1.22\lambda/d$

**Photons and light:**

$$\lambda \nu = c$$

$$E_{ph} = h\nu = cp_{ph}$$

$$p_{ph} = \frac{h}{\lambda}$$

Photoelectric effect:  $K = h\nu - \phi = eV_s$   
 Where  $K$  is the kinetic energy of the emitted electrons,  $\phi$  is the work function of the material and  $V_s$  is the stopping potential.

Black body radiation:

$$I = \sigma T^4$$

$$\lambda_{max} T = 2.898 \times 10^{-3} m \cdot K$$

$$u(\lambda) = \left( \frac{8\pi hc}{\lambda^5} \right) \left[ \frac{1}{e^{hc/\lambda kT} - 1} \right]$$

$$dI = \frac{c}{4} u(\lambda) d\lambda$$

Where  $u(\lambda)$  is the energy density,  $h$  is Planck's constant,  $k$  is the Boltzmann constant and  $c$  is the speed of light.

Compton Scattering:  $\lambda' - \lambda = \frac{h}{m_e c} (1 - \cos \theta)$

Bremsstrahlung:  $\lambda_{min} = \frac{hc}{eV}$

**Wavelike properties of particles:**

De Broglie wavelength:  $\lambda = \frac{h}{p}$   
 Hiesenberg uncertainty relationships:

$$\Delta E \Delta t \geq \frac{\hbar}{2}$$

$$\Delta p_x \Delta x \geq \frac{\hbar}{2}$$

$$\Delta x = \left( \langle x^2 \rangle - \langle x \rangle^2 \right)^{1/2}$$

$$\Delta p = \left( \langle p^2 \rangle - \langle p \rangle^2 \right)^{1/2}$$

Wave packets:

$$p = h / \lambda = \hbar k$$

$$\hbar \omega = \frac{p^2}{2m} = \frac{\hbar^2 k^2}{2m}$$

$$v_{group} = \frac{d\omega}{dk}$$

$$v_{phase} = \frac{\omega}{k}$$

**Quantum Mechanics:**

Schrödinger equation:

Complete:  $-\frac{\hbar^2}{2m} \frac{\partial^2 \Psi(x,t)}{\partial x^2} + V(x,t) \Psi(x,t) = i\hbar \frac{\partial \Psi(x,t)}{\partial t}$

Time independent:  $-\frac{\hbar^2}{2m} \frac{\partial^2 \Psi(x)}{\partial x^2} + V(x) \Psi(x) = E \Psi(x)$

Probability Current:  $S(x,t) = \frac{i\hbar}{2m} \left\{ \Psi \frac{\partial \Psi^*}{\partial x} - \Psi^* \frac{\partial \Psi}{\partial x} \right\}$

Normalization (1-D):  $\int_{-\infty}^{+\infty} \Psi^* \Psi dx = 1$

Normalization (3-D):  $\int_0^{+\infty} \int_0^\pi \int_0^{2\pi} \Psi^* \Psi r^2 \sin \theta dr d\theta d\phi = 1$  or  $\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \Psi^* \Psi dx dy dz = 1$

Momentum operator:  $i\hbar \frac{\partial}{\partial x}$       Energy operator:  $-i\hbar \frac{\partial}{\partial t}$

**Nuclear Physics:**

$$R = R_0 A^{1/3} = 1.2 A^{1/3} \text{ fm}$$

$$B = \left[ Zm \left( {}^1_1H \right) + Nm_n - m \left( {}^A_ZX \right) \right] c^2$$

$$Q = [M_{parent} - M_{Daughter} - M_{emitted}] c^2$$

$$N = N_0 e^{-\lambda t}$$

$$A = \lambda N$$

$$t_{1/2} = \frac{\ln 2}{\lambda} = \frac{0.693}{\lambda}$$

**Nuclear Atom:**

$$F = \frac{(ze)(Ze)}{4\pi\epsilon_0 r^2}$$

$$U = \frac{(ze)(Ze)}{4\pi\epsilon_0 r}$$

Rutherford Scattering:

$$b = \frac{zZ}{2K} \frac{e^2}{4\pi\epsilon_0} \cot \frac{1}{2} \theta$$

$$\frac{1}{2} m v^2 = \frac{1}{2} \left( \frac{b^2 v^2}{r_{min}^2} \right) + \frac{e^2}{4\pi\epsilon_0} \frac{zZ}{r_{min}}$$

$$d = \frac{1}{4\pi\epsilon_0} \frac{zZe^2}{K}$$

Bohr model:  $E_n = -\frac{me^4}{32\pi^2 \epsilon_0^2 \hbar^2} \frac{1}{n^2} = -\frac{13.6 Z_{eff}^2}{n^2} eV$

$$\frac{1}{\lambda} = R \left( \frac{1}{n_f^2} - \frac{1}{n_i^2} \right)$$

$$R = R_\infty \left( \frac{1}{1 + m/M} \right)$$

$$R_\infty = \frac{mk^2 e^4}{4\pi c \hbar^3} \text{ where } k = \frac{1}{4\pi\epsilon_0}$$

Bohr radius:  $a_0 = \frac{\hbar^2}{4\pi\epsilon_0 m e^2}$

$$r_n = n^2 a_0 / Z$$

**X-rays:**

K-series:  $E_{photon} = (13.6eV) \left( \frac{1}{1^2} - \frac{1}{n^2} \right) (Z-1)^2$

L-series:  $E_{photon} = (13.6eV) \left( \frac{1}{2^2} - \frac{1}{n^2} \right) (Z-3)^2$

M-series:  $E_{photon} = (13.6eV) \left( \frac{1}{3^2} - \frac{1}{n^2} \right) (Z-5)^2$

**Some useful mathematical relations:**

$$\sqrt{\left(1 - \frac{u^2}{c^2}\right)} \approx 1 - \frac{1}{2} \frac{u^2}{c^2}$$

$$\frac{1}{\sqrt{\left(1 - \frac{u^2}{c^2}\right)}} \approx 1 + \frac{1}{2} \frac{u^2}{c^2}$$

For  $u^2/c^2 \ll 1$

**Some useful trigonometric relations:**

$$\sin(A + B) = \sin(A) \cos(B) + \cos(A) \sin(B)$$

$$\cos(A + B) = \cos(A) \cos(B) - \sin(A) \sin(B)$$

$$\sin(2A) = 2 \sin(A) \cos(A)$$

$$\sin^2(A) = \frac{1 - \cos(2A)}{2}$$

$$\cos^2(A) = \frac{1 + \cos(2A)}{2}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

**Some useful Integrals:**

$$\int \sin^2(u) du = \frac{1}{2}(u - \sin u \cos u)$$

$$\int \sin u \cos u du = \frac{1}{2} \sin^2 u$$

$$\int \cos^2 u du = \frac{1}{2}(u + \sin u \cos u)$$

$$\int u \sin^2 u du = \frac{u^2}{4} - \frac{u \sin 2u}{4} - \frac{\cos 2u}{8}$$

$$\int u \cos^2 u du = \frac{u^2}{4} + \frac{u \sin 2u}{4} + \frac{\cos 2u}{8}$$

$$\int u^2 \sin^2 u du = \frac{u^3}{6} - \left(\frac{u^2}{4} - \frac{1}{8}\right) \sin 2u - \frac{u \cos 2u}{4}$$

$$\int u^2 \cos^2 u du = \frac{u^3}{6} + \left(\frac{u^2}{4} - \frac{1}{8}\right) \sin 2u + \frac{u \cos 2u}{4}$$

$$\int \sin^3 u du = -\frac{1}{3} \cos u [\sin^2 u + 2]$$

$$\int \cos^3 u du = \frac{1}{3} \sin u [\cos^2 u + 2]$$

$$\int_0^\infty u^n e^{-u} du = n! \text{ for } n > 0$$

$$\int \cos^n u \sin u du = -\frac{\cos^{n+1} u}{n+1} \text{ for } n > 0$$

$$\int \sin^n u \cos u du = \frac{\sin^{n+1} u}{n+1} \text{ for } n > 0$$

$$\int_0^\infty \frac{z^3}{e^z - 1} dz = \frac{\pi^4}{15}$$

If n is even	$\int_{-\infty}^\infty x^n e^{-\lambda x^2} dx = 2I_n$ ( $I_n$ defined below)
If n is odd	$\int_{-\infty}^\infty x^n e^{-\lambda x^2} dx = 0$
<b>n</b>	$I_n = \int_0^\infty x^n e^{-\lambda x^2} dx$
0	$\frac{1}{2} \pi^{1/2} \lambda^{-1/2}$
1	$\frac{1}{2} \lambda^{-1}$
2	$\frac{1}{4} \pi^{1/2} \lambda^{-3/2}$
3	$\frac{1}{2} \lambda^{-2}$
4	$\frac{3}{8} \pi^{1/2} \lambda^{-5/2}$
5	$\lambda^{-3}$

**Constants:**

Constant	Standard value	Alternate units
Speed of light	$c = 2.998 \times 10^8 \text{ m/s}$	
Electronic charge	$e = 1.602 \times 10^{-19} \text{ C}$	
Boltzmann constant	$k = 1.381 \times 10^{-23} \text{ J/K}$	$8.617 \times 10^{-5} \text{ eV/K}$
Planck's constant	$h = 6.626 \times 10^{-34} \text{ J}\cdot\text{s}$	$4.136 \times 10^{-15} \text{ eV}\cdot\text{s}$
	$\hbar = 1.055 \times 10^{-34} \text{ J}\cdot\text{s}$	$0.652 \times 10^{-15} \text{ eV}\cdot\text{s}$
Avogadro's constant	$N_A = 6.022 \times 10^{23} \text{ mole}^{-1}$	
Stefan-Boltzmann constant	$\sigma = 5.67 \times 10^{-8} \text{ W/m}^2 \cdot \text{K}^4$	
Electron mass	$m_e = 5.49 \times 10^{-4} \text{ u or } 9.109 \times 10^{-31} \text{ kg}$	$0.511 \text{ MeV}/c^2$
Proton mass	$1.007276 \text{ u or } 1.673 \times 10^{-27} \text{ kg}$	$938.3 \text{ MeV}/c^2$
Neutron mass	$1.008665 \text{ u or } 1.675 \times 10^{-27} \text{ kg}$	$939.6 \text{ MeV}/c^2$
Mass of <sup>4</sup> He	$4.002603 \text{ u}$	
Bohr radius	$a_0 = 4\pi\epsilon_0 \hbar^2 / m_e e^2 = 0.0529 \text{ nm}$	
Hydrogen ionization energy	$13.6 \text{ eV}$	
	$hc = 1240 \text{ eV}\cdot\text{nm}$	
	$1 \text{ eV} = 1.602 \times 10^{-19} \text{ J}$	
Atomic mass unit (dalton)	$1 \text{ u} = 931.5 \text{ MeV}/c^2$	$1.661 \times 10^{-27} \text{ kg}$
	$kT = 0.02525 \text{ eV} \approx \frac{1}{40} \text{ eV}$ at T=293 K	
	$\frac{1}{4\pi\epsilon_0} = 8.988 \times 10^9 \text{ N}\cdot\text{m}^2 \cdot \text{C}^{-2}$	
	$\frac{e^2}{4\pi\epsilon_0} = 1.44 \text{ eV}\cdot\text{nm}$	