UNIVERSITY OF MANITOBA

April 20th, 2010 **Final Exam**

NO. OF PAGES: 7

LOCATION: 405 Allen Bldg.

DEPT. AND COURSE NO.: PHYS 2380 **TIME: 09:00 to 12:00 (3 hours)**

EXAMINATION: Quantum Mechanics 1 **Examiners: K.S. Sharma**

Instructions:

- Answer all questions.
- Start each question on a new page.
- Show all of your steps in arriving at the solution. Do not just state the answer. State your arguments clearly and indicate any assumptions you make.
- Include a sentence or two at the end of each problem summarizing your solution.
- Each part of a question carries equal weight.

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1. The energy density, as a function of wavelength, λ, and temperature, T, inside a black body is given by $u(\lambda) = \frac{8\pi hc}{\lambda^5}\frac{1}{e^{hc/\lambda kT} - 1}$ where *h* is Planck's constant, *k* is the Boltzmann constant and *c* is the speed of light.

   a) Differentiate the expression for $u(\lambda)$ and find the wavelength, λ_m, where the intensity of the emitted light is the greatest?
   b) Transform the expression for $u(\lambda)$ into an expression for the energy density as a function of frequency, $u(\nu)$? Recall that the definition of energy density means that the amount of energy contained in a range of wavelengths $d\lambda$ is given by $u(\lambda)\, d\lambda$ and that your results for $u(\nu)d\nu$ must be consistent with this.
   c) Differentiate the expression in part (b) and find the frequency, ν_m, where the emitted light is greatest.
   d) Is the product $\lambda_m\ \nu_m = c$? Comment on your result.

   Hint: You may assume that $e^{hc/\lambda kT} \gg 1$ at λ_m to simplify the procedure.

   (15 marks)

2. A photon with a wavelength of 0.0711 nm are scattered from an electron at an angle of 180° to its original direction. Express your answers in eV and nanometers.
   a) What is the original energy of this photon?
   b) What is the energy and wavelength of the photon after it was scattered.
   c) What is the recoil energy and de Broglie wavelength of the scattered electron.

   (12 marks)

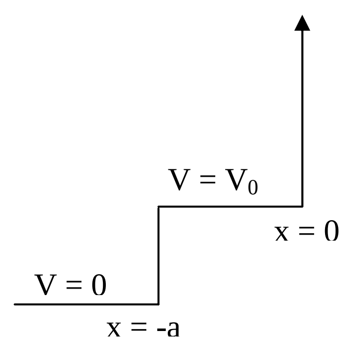
3. An electron is confined in an infinite square well potential of width *a*. You may find it useful to consider such a potential as extending over the region $0 \le x \le a$.

   a) Solve the S. equation for the wave functions of such a particle and derive an expression for the allowed values of its energy.
   b) Calculate the average value for x (where x is the coordinate of the trapped particle).
   c) Calculate the average value for x^2 (where x is the coordinate of the trapped particle).
   d) From these calculate the uncertainty in the position of the particle, Δx.

   (20 marks)

4. For the potential function, V(x), shown here:

   a) Write down the “space-part” wave function for a particle with energy $E < V_0$ in region 1 ($x < -a$), region 2 ($-a < x < 0$) and region 3 ($x > 0$). Provide expressions for the wave number (k) in each region. Identify the incident and reflected components in these wave functions.



   b) What are the corresponding “time-parts” of the wave function?
   c) Apply the boundary conditions and determine the amplitudes for all the component waves in terms of one of them.
   d) From the coefficients determined in part (c) show that the probability current is zero in all three regions. Comment on this outcome.

   (20 marks)

(Continued on following page)

5. The normalized wave functions for the n=2 state of the hydrogen atom are:

$$\psi_{nlm} = R_{nl}\left(r\right)\Theta_{lm}\left(\theta\right)\Theta\left(\phi\right)\ where:$$

$$R_{20} = \frac{1}{\sqrt{2a_0^3}}\left(1 - \frac{r}{2a_0}\right)e^{-r/2a_0}$$

$$R_{21} = \frac{1}{2\sqrt{6a_0^3}}\left(\frac{r}{a_0}\right)e^{-r/2a_0}$$

$$and\ the\ angular\ dependence\ is\ given\ by:$$

$$\Theta_{11}\Phi_1 = \sqrt{\frac{3}{8\pi}}\sin\theta e^{i\phi}$$

$$\Theta_{10}\Phi_0 = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$\Theta_{11}\Phi_{-1} = \sqrt{\frac{3}{8\pi}}\sin\theta e^{-i\phi}$$

where a_0 is the Bohr radius.

a) Calculate the expectation value for r for R_{20} and compare it with the radius predicted by Bohr's theory for this energy level.
b) Sketch a polar diagram representing the modulation factors that result from the 3 possibilities for the angular wave functions. Comment on the shapes of these diagrams and what they tell us about the orientation of the planes of the orbits.
c) Is R_{21} normalized by itself?
d) Is $\Theta_{11}\Phi_1$ normalized by itself?

(20 marks)

The End

Appendix: Some information from the text and lectures:

Special Relativity:

Relativistic momentum and energy:

$$\vec{p} = \gamma m\vec{v}$$
$$E = \gamma mc^2 = mc^2 + K$$
$$E^2 = c^2 p^2 + m^2 c^4$$

Electromagnetic radiation:

Power received by a detector from a wave:

$$P = \left(\frac{1}{\mu_0 c}\right) E_0^2 A \sin^2(kz - \omega t + \phi)$$
$$P_{ave} = \frac{1}{T_0}\int_0^{T_0} P\,dt = \frac{E_0^2 A}{2\mu_0 c}, \quad I = \frac{P_{ave}}{A} = \frac{E_0^2}{2\mu_0 c}$$

Where P is the instantaneous power, P_{ave} is the average power delivered to a detector of area A and I is the intensity of the light.

Interference and diffraction:

Pattern Type	Bright Fringes	Dark Fringes
Single slit (width *w*)	$\frac{w}{2}\sin\theta = m\lambda$	$\frac{w}{2}\sin\theta = \left(m + \frac{1}{2}\right)\lambda$
Double slit (spacing *d*)	$d\sin\theta = m\lambda$	
Grating (lines spaced *d* apart)	$d\sin\theta = m\lambda$	
Bragg (layers of atoms *d* apart)	$2d\sin\theta = m\lambda$	
Circular object		First fringe at $1.22\lambda/d$

Photons and light:

$$\lambda\nu = c$$
$$E_{ph} = h\nu = cp_{ph}$$
$$p_{ph} = h/\lambda$$

Photoelectric effect: $K = h\nu - \phi = eV_s$

Where K is the kinetic energy of the emitted electrons, ϕ is the work function of the material and V_s is the stopping potential.

Black body radiation:

$$I = \sigma T^4$$
$$\lambda_{\max} T = 2.898\times10^{-3}\, m \bullet K$$
$$u(\lambda) = \left(\frac{8\pi hc}{\lambda^5}\right)\left[\frac{1}{e^{hc/\lambda kT} - 1}\right]$$
$$dI = \frac{c}{4}u(\lambda)\,d\lambda$$

Where *u(λ)* is the energy density, *h* is Planck's constant, *k* is the Botzmann constant and *c* is the speed of light.

Compton Scattering: $\lambda' - \lambda = \frac{h}{m_e c}(1 - \cos\theta)$

Bremsstrahlung: $\lambda_{\min} = \frac{hc}{eV}$

Wavelike properties of particles:

De Broglie wavelength: $\lambda = h/p$

Hiesenberg uncertainty relationships:

$$\Delta E\Delta t \ge \frac{\hbar}{2} \qquad \Delta x = \left(\langle x^2\rangle - \langle x\rangle^2\right)^{1/2}$$
$$\Delta p_x \Delta x \ge \frac{\hbar}{2} \qquad \Delta p = \left(\langle p^2\rangle - \langle p\rangle^2\right)^{1/2}$$

Wave packets:

$$v_{group} = \frac{d\omega}{dk}$$

$$v_{phase} = \frac{\omega}{k}$$

$$p = h/\lambda = \hbar k$$

$$\hbar\omega = \frac{p^2}{2m} = \frac{\hbar^2 k^2}{2m}$$

Quantum Mechanics:

Schrödinger equation:

Complete: $-\frac{\hbar^2}{2m}\frac{\partial^2\Psi(x,t)}{\partial x^2} + V(x,t)\Psi(x,t) = i\hbar\frac{\partial\Psi(x,t)}{\partial t}$

Time independent: $-\frac{\hbar^2}{2m}\frac{\partial^2\psi(x)}{\partial x^2} + V(x)\psi(x) = E\psi(x)$

Probability Current: $S(x,t) = \frac{i\hbar}{2m}\left\{\Psi\frac{\partial\Psi^*}{\partial x} - \Psi^*\frac{\partial\Psi}{\partial x}\right\}$

Normalization (1-D): $\int_{-\infty}^{+\infty}\Psi^*\Psi dx = 1$

Normalization (3-D): $\int_0^{+\infty}\int_0^{\pi}\int_0^{2\pi}\Psi^*\Psi\, r^2\sin\theta\, dr\, d\theta\, d\phi = 1$ or $\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\Psi^*\Psi\, dx\, dy\, dz = 1$

Nuclear Physics:

$$R = R_0 A^{1/3} = 1.2A^{1/3}\, fm$$

$$B = \left[Zm\left({}_1^1H\right) + Nm_n - m\left({}_z^AX\right)\right]c^2$$

$$Q = [M_{parent} - M_{Daughter} - M_{emitted}]c^2$$

$$N = N_0 e^{-\lambda t}$$

$$A = \lambda N$$

$$t_{1/2} = \frac{\ln 2}{\lambda} = \frac{0.693}{\lambda}$$

Nuclear Atom:

$$F = \frac{(ze)(Ze)}{4\pi\varepsilon_0 r^2}$$

$$U = \frac{(ze)(Ze)}{4\pi\varepsilon_0 r}$$

Rutherford Scattering:

$$b = \frac{zZ}{2K}\frac{e^2}{4\pi\varepsilon_0}\cot\tfrac{1}{2}\theta$$

$$\tfrac{1}{2}mv^2 = \tfrac{1}{2}\left(\frac{b^2v^2}{r_{\min}^2}\right) + \frac{e^2}{4\pi\varepsilon_0}\frac{zZ}{r_{\min}}$$

$$d = \frac{1}{4\pi\varepsilon_0}\frac{zZe^2}{K}$$

Bohr model: $E_n = -\frac{me^4}{32\pi^2\varepsilon_0^2\hbar^2}\frac{1}{n^2} = -\frac{13.6Z_{eff}^2}{n^2}eV$

$$\frac{1}{\lambda} = R\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right)$$

$$R = R_\infty\left(\frac{1}{1+m/M}\right)$$

$$R_\infty = \frac{mk^2e^4}{4\pi c\hbar^3} \text{ where } k = \frac{1}{4\pi\varepsilon_0}$$

Bohr radius: $a_0 = \frac{\hbar^2}{4\pi\varepsilon_0 me^2}$

$$r_n = n^2 a_0 / Z$$

X-rays:

K-series: $E_{photon} = (13.6eV)\left(\frac{1}{1^2} - \frac{1}{n^2}\right)(Z-1)^2$

L-series: $E_{photon} = (13.6eV)\left(\frac{1}{2^2} - \frac{1}{n^2}\right)(Z-3)^2$

M-series: $E_{photon} = (13.6eV)\left(\frac{1}{3^2} - \frac{1}{n^2}\right)(Z-5)^2$

Some useful mathematical relations:

$$\sqrt{\left(1-u^2/c^2\right)} \approx 1-\frac{1}{2}\frac{u^2}{c^2}$$

$$\frac{1}{\sqrt{\left(1-u^2/c^2\right)}} \approx 1+\frac{1}{2}\frac{u^2}{c^2}$$

For $u^2/c^2 \ll 1$

Some useful trigonometric relations:

$$\sin(A+B) = \sin(A)\cos(B) + \cos(A)\sin(B)$$

$$\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$$

$$\sin(2A) = 2\sin(A)\cos(A)$$

$$\sin^2(A) = \frac{1-\cos(2A)}{2}$$

$$\cos^2(A) = \frac{1+\cos(2A)}{2}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Some useful Integrals:

$$\int \sin^2(u)\,du = \frac{1}{2}\left(u - \sin u \cos u\right)$$

$$\int \sin u \cos u\,du = \frac{1}{2}\sin^2 u$$

$$\int \cos^2 u\,du = \frac{1}{2}\left(u + \sin u \cos u\right)$$

$$\int u\sin^2 u\,du = \frac{u^2}{4} - \frac{u\sin 2u}{4} - \frac{\cos 2u}{8}$$

$$\int u\cos^2 u\,du = \frac{u^2}{4} + \frac{u\sin 2u}{4} + \frac{\cos 2u}{8}$$

$$\int u^2\sin^2 u\,du = \frac{u^3}{6} - \left(\frac{u^2}{4} - \frac{1}{8}\right)\sin 2u - \frac{u\cos 2u}{4}$$

$$\int u^2\cos^2 u\,du = \frac{u^3}{6} + \left(\frac{u^2}{4} - \frac{1}{8}\right)\sin 2u + \frac{u\cos 2u}{4}$$

$$\int \sin^3 u\,du = -\frac{1}{3}\cos u\left[\sin^2 u + 2\right]$$

$$\int \cos^3 u\,du = \frac{1}{3}\sin u\left[\cos^2 u + 2\right]$$

$$\int_0^\infty u^n e^{-u}\,du = n! \quad \mathit{for}\ n>0$$

$$\int \cos^n u \sin u\,du = -\frac{\cos^{n+1} u}{n+1} \quad \mathit{for}\ n>0$$

$$\int \sin^n u \cos u\,du = \frac{\sin^{n+1} u}{n+1} \quad \mathit{for}\ n>0$$

n	$I_n = \int_0^\infty x^n e^{-\lambda x^2}\,dx$
0	$\frac{1}{2}\pi^{1/2}\lambda^{-1/2}$
1	$\frac{1}{2}\lambda^{-1}$
2	$\frac{1}{4}\pi^{1/2}\lambda^{-3/2}$
3	$\frac{1}{2}\lambda^{-2}$

n	$I_n = \int_0^\infty x^n e^{-\lambda x^2}\,dx$
4	$\frac{3}{8}\pi^{1/2}\lambda^{-5/2}$
5	λ^{-3}
If n is even	$\int_{-\infty}^{\infty} x^n e^{-\lambda x^2}\,dx = 2I_n$
If n is odd	$\int_{-\infty}^{\infty} x^n e^{-\lambda x^2}\,dx = 0$

Constants:

Constant	Standard value	Alternate units
Speed of light	$c = 2.998\times10^{8} m/s$	
Electronic charge	$e = 1.602\times10^{-19} C$	
Boltzmann constant	$k = 1.381\times10^{-23} J/K$	$8.617\times10^{-5} eV/K$
Planck's constant	$h = 6.626\times10^{-34} J\cdot s$	$4.136\times10^{-15} eV\cdot s$
	$\hbar = 1.055\times10^{-34} J\cdot s$	$0.652\times10^{-15} eV\cdot s$
Avogadro's constant	$N_A = 6.022\times10^{23} mole^{-1}$	
Stefan-Boltzmann constant	$\sigma = 5.67\times10^{-8}\ W/m^2\cdot K^4$	
Electron mass	$m_e = 5.49\times10^{-4} u\ or\ 9.109\times10^{-31}\ kg$	$0.511 MeV/c^2$
Proton mass	$1.007276\ u\ or\ 1.673\times10^{-27}\ kg$	$938.3 MeV/c^2$
Neutron mass	$1.008665\ u\ or\ 1.675\times10^{-27}\ kg$	$939.6 MeV/c^2$
Mass of ^{4}He	*4.002603 u*	
Bohr radius	$a_0 = 4\pi\varepsilon_0\hbar^2 / m_e e^2 = 0.0529 nm$	
Hydrogen ionization energy	$13.6 eV$	
	$hc = 1240 ev\cdot nm$	
	$1 eV = 1.602\times10^{-19} J$	
Atomic mass unit (dalton)	$1u = 931.5 MeV/c^2$	$1.661\times10^{-27} kg$
	$kT = 0.02525 eV \approx \frac{1}{40} eV$ at T=293 K	
	$\frac{1}{4\pi\varepsilon_0} = 8.988\times10^{9} N\cdot m^2\cdot C^{-2}$	
	$\frac{e^2}{4\pi\varepsilon_0} = 1.44 eV\cdot nm$	