UNIVERSITY OF MANITOBA

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Final Exam

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PAPER NO.: 272

DEPT. AND COURSE NO.: PHYS 2380

TIME: 3 hours

EXAMINATION: Quantum Mechanics 1

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Instructions:

- Answer all questions.
- Start each question on a new page.
- Show all of your steps in arriving at the solution. Do not just state the answer. State your arguments clearly and indicate any assumptions you make.
- Include a sentence or two at the end of each problem summarizing your solution.
- Each part of a question carries equal weight.

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1. The spectral radiancy for a black body is given by $\rho_T(\lambda) = \frac{8\pi hc}{\lambda^5}\frac{1}{e^{hc/\lambda kT}-1}$. Derive the Wien displacement law: $\lambda_{\max} T \approx 0.20hc/k$ (where k is the Boltzmann constant) by solving the equation $d\rho_T(\lambda)/d\lambda = 0$ for the wavelength, $\lambda_{\max}$, where the radiancy is a maximum. Explain any approximations you make. (10 marks)

2. A high energy electron beam is scattered by carbon (A = 12) nucleus. A measurement shows that the first minimum in the diffraction pattern occurs at an angle of 50°.

   a) What is the diameter of the nucleus of the carbon atom?
   b) What is the momentum of the electron?
   c) What is the energy of the electron? Consider if this is a relativistic or non-relativistic problem.

   (10 marks)

3. A particle is confined in a potential well such that:
   V(x) = 0 for 0 < x< a and
   V(x) = ∞ everywhere else.

   a) Find the normalized eigenfunctions for the time-independent Schrödinger equation.
   b) Find the energies for the allowed states.
   c) Given that the expectation value for the momentum is given by $\langle p \rangle = 0$, determine $\Delta p = \left(\langle p^2 \rangle - \langle p \rangle^2\right)^{\frac{1}{2}}$.

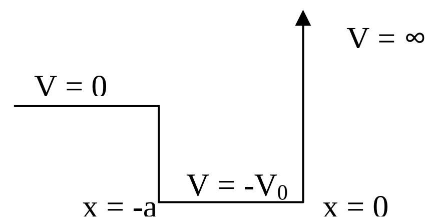
d) Given that the expectation value for position is given by $\langle x \rangle = a/2$,

determine $\Delta x = \left(\langle x^2 \rangle - \langle x \rangle^2 \right)^{\frac{1}{2}}$.

e) Does the product $\Delta x \Delta p$ satisfy the requirements of the Hiesenberg uncertainty principle?

(25 marks)

4. For the potential function shown :



a) Write down the wave function for a particle with energy E > 0 in region 1 (x < -a) and region 2 (-a < x < 0). Provide expressions for the wave number (k) in each region. Identify the incident and reflected components in these wave functions.

b) Apply the boundary conditions and determine the amplitudes for all the component waves in terms of one of them.

c) From the coefficients determined in part (b) show that the reflection coefficient in region 1 (x < -a) is 1 (you should not have to do a painful algebraic exercise to do this).

(15 marks)

5. The normalized wave function for the n=3, l=2, m=0 state of the hydrogen atom has the form:

$$\psi_{320} = \frac{1}{81\sqrt{6\pi}} \left(\frac{1}{a_0} \right)^{\frac{3}{2}} \left(\frac{1}{a_0^2} \right) r^2 e^{-r/3a_0} \left(3\cos^2\theta - 1 \right) e^{i\phi}$$

a) Calculate the expectation value for r, <r>

b) Differentiate the radial probability density function and determine the radius at which it is a maximum.

c) Explain why the answers to parts (a) and (b) are not the same?

(15 marks)

The End

Appendix: Some information from the text and lectures:

Special Relativity:

Relativistic momentum and energy:

$$\vec{p} = \gamma m \vec{v}$$
$$E = \gamma mc^2 = mc^2 + K$$
$$E^2 = c^2 p^2 + m^2 c^4$$

Electromagnetic radiation:

Power received by a detector from a wave:

$$P = \left(\frac{1}{\mu_0 c}\right) E_0^2 A \sin^2(kz - \omega t + \phi)$$
$$P_{ave} = \frac{1}{T_0}\int_0^{T_0} P dt = \frac{E_0^2 A}{2\mu_0 c}, \quad I = \frac{P_{ave}}{A} = \frac{E_0^2}{2\mu_0 c}$$

Where P is the instantaneous power, P_{ave} is the average power delivered to a detector of area A and I is the intensity of the light.

Interference and diffraction:

Pattern Type	Bright Fringes	Dark Fringes
Single slit (width *w*)	$\frac{w}{2}\sin\theta = m\lambda$	$\frac{w}{2}\sin\theta = \left(m + \frac{1}{2}\right)\lambda$
Double slit (spacing *d*)	$d\sin\theta = m\lambda$	
Grating (lines spaced *d* apart)	$d\sin\theta = m\lambda$	
Bragg (layers of atoms *d* apart)	$2d\sin\theta = m\lambda$	
Circular object		First fringe at $1.22\lambda/d$

Photons and light:

$$\lambda \nu = c$$
$$E_{ph} = h\nu = cp_{ph}$$
$$p_{ph} = h/\lambda$$

Photoelectric effect: $K = h\nu - \phi = eV_s$

Where K is the kinetic energy of the emitted electrons, ϕ is the work function of the material and V_s is the stopping potential.

Black body radiation:

$$I = \sigma T^4$$
$$\lambda_{\max} T = 2.898\times 10^{-3} m \bullet K$$
$$\rho_T(\lambda) = \left(\frac{2\pi hc^2}{\lambda^5}\right)\left[\frac{1}{e^{hc/\lambda kT} - 1}\right]$$
$$dI = \rho_T(\lambda) d\lambda$$

Compton Scattering: $\lambda' - \lambda = \frac{h}{m_e c}(1 - \cos\theta)$

Bremsstrahlung: $\lambda_{\min} = \frac{hc}{eV}$

Wavelike properties of particles:

De Broglie wavelength: $\lambda = h/p$

Hiesenberg uncertainty relationships:

$$\Delta E \Delta t \geq \frac{\hbar}{2}$$
$$\Delta p_x \Delta x \geq \frac{\hbar}{2}$$

Wave packets:

$$p = h/\lambda = \hbar k$$

$$\hbar\omega = \frac{p^2}{2m} = \frac{\hbar^2 k^2}{2m}$$

$$v_{group} = \frac{d\omega}{dk}$$

$$v_{phase} = \frac{\omega}{k}$$

Quantum Mechanics:

Schrödinger equation:

Complete: $-\frac{\hbar^2}{2m}\frac{\partial^2\Psi(x,t)}{\partial x^2} + V(x,t)\Psi(x,t) = i\hbar\frac{\partial\Psi(x,t)}{\partial t}$

Time independent: $-\frac{\hbar^2}{2m}\frac{\partial^2\psi(x)}{\partial x^2} + V(x)\psi(x) = E\psi(x)$

Probability Current: $S(x,t) = \frac{i\hbar}{2m}\left\{\Psi\frac{\partial\Psi^*}{\partial x} - \Psi^*\frac{\partial\Psi}{\partial x}\right\}$

Normalization (1-D): $\int_{-\infty}^{+\infty}\Psi^*\Psi dx = 1$

Normalization (3-D): $\int_0^{+\infty}\int_0^{\pi}\int_0^{2\pi}\Psi^*\Psi\, r^2\sin\theta\, dr\, d\theta\, d\phi = 1$ or $\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\Psi^*\Psi\, dx\, dy\, dz = 1$

Nuclear Physics:

$$R = R_0 A^{1/3} = 1.2A^{1/3}\, fm$$

$$B = \left[Zm\left({}^1_1H\right) + Nm_n - m\left({}^A_zX\right)\right]c^2$$

$$Q = [M_{parent} - M_{Daughter} - M_{emitted}]c^2$$

$$N = N_0 e^{-\lambda t}$$

$$A = \lambda N$$

$$t_{1/2} = \frac{\ln 2}{\lambda} = \frac{0.693}{\lambda}$$

Nuclear Atom:

$$F = \frac{(ze)(Ze)}{4\pi\varepsilon_0 r^2}$$

$$U = \frac{(ze)(Ze)}{4\pi\varepsilon_0 r}$$

Rutherford Scattering:

$$b = \frac{zZ}{2K}\frac{e^2}{4\pi\varepsilon_0}\cot\tfrac{1}{2}\theta$$

$$\tfrac{1}{2}mv^2 = \tfrac{1}{2}\left(\frac{b^2v^2}{r_{\min}^2}\right) + \frac{e^2}{4\pi\varepsilon_0}\frac{zZ}{r_{\min}}$$

$$d = \frac{1}{4\pi\varepsilon_0}\frac{zZe^2}{K}$$

Bohr model: $E_n = -\frac{me^4}{32\pi^2\varepsilon_0^2\hbar^2}\frac{1}{n^2} = -\frac{13.6Z_{eff}^2}{n^2}eV$

X-rays:

K-series: $E_{photon} = (13.6eV)\left(\frac{1}{1^2} - \frac{1}{n^2}\right)(Z-1)^2$

L-series: $E_{photon} = (13.6eV)\left(\frac{1}{2^2} - \frac{1}{n^2}\right)(Z-3)^2$

M-series: $E_{photon} = (13.6eV)\left(\frac{1}{3^2} - \frac{1}{n^2}\right)(Z-5)^2$

Some useful mathematical relations:

$$\sqrt{\left(1-u^2/c^2\right)} \approx 1-\frac{1}{2}\frac{u^2}{c^2}$$

$$\frac{1}{\sqrt{\left(1-u^2/c^2\right)}} \approx 1+\frac{1}{2}\frac{u^2}{c^2}$$

For $u^2/c^2 << 1$

$$\sin(A+B) = \sin(A)\cos(B) + \cos(A)\sin(B)$$

$$\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$$

$$\sin(2A) = 2\sin(A)\cos(A)$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Some useful Integrals:

$$\int \sin^2(u)\,du = \frac{1}{2}\left(u - \sin u \cos u\right)$$

$$\int \sin u \cos u\,du = \frac{1}{2}\sin^2 u$$

$$\int \cos^2 u\,du = \frac{1}{2}\left(u + \sin u \cos u\right)$$

$$\int u \sin^2 u\,du = \frac{u^2}{4} - \frac{u\sin 2u}{4} - \frac{\cos 2u}{8}$$

$$\int u \cos^2 u\,du = \frac{u^2}{4} + \frac{u\sin 2u}{4} + \frac{\cos 2u}{8}$$

$$\int u^2 \sin^2 u\,du = \frac{u^3}{6} - \left(\frac{u^2}{4} - \frac{1}{8}\right)\sin 2u - \frac{u\cos 2u}{4}$$

$$\int u^2 \cos^2 u\,du = \frac{u^3}{6} + \left(\frac{u^2}{4} - \frac{1}{8}\right)\sin 2u + \frac{u\cos 2u}{4}$$

$$\int_0^\infty u^n e^{-u}\,du = n! \quad for\ n > 0$$

$$\int \cos^n u \sin u\,du = -\frac{\cos^{n+1} u}{n+1} \quad for\ n > 0$$

$$\int \sin^n u \cos u\,du = \frac{\sin^{n+1} u}{n+1} \quad for\ n > 0$$

Constants:

Constant	Standard value	Alternate units
Speed of light	$c = 2.998\times10^{8} m/s$	
Electronic charge	$e = 1.602\times10^{-19} C$	
Boltzmann constant	$k = 1.381\times10^{-23} J/K$	$8.617\times10^{-5} eV/K$
Planck's constant	$h = 6.626\times10^{-34} J\cdot s$	$4.136\times10^{-15} eV\cdot s$
	$\hbar = 1.055\times10^{-34} J\cdot s$	$0.652\times10^{-15} eV\cdot s$
Avogadro's constant	$N_A = 6.022\times10^{23} mole^{-1}$	
Stefan-Boltzmann constant	$\sigma = 5.67\times10^{-8}\ W/m^2\cdot K^4$	
Electron mass	$m_e = 5.49\times10^{-4} u$	$0.511 MeV/c^2$
Proton mass	$1.007276\ u$	$938.3 MeV/c^2$
Neutron mass	$1.008665\ u$	$939.6 MeV/c^2$
Mass of ^{4}He	*4.002603 u*	
Bohr radius	$a_0 = 4\pi\varepsilon_0\hbar^2 / m_e e^2 = 0.0529 nm$	
Hydrogen ionization energy	$13.6 eV$	
	$hc = 1240 ev\cdot nm$	
	$1eV = 1.602\times10^{-19} J$	
Atomic mass unit (dalton)	$1u = 931.5 MeV/c^2$	$1.661\times10^{-27} kg$
	$kT = 0.02525 eV \approx \frac{1}{40} eV$ at T=293 K	
	$\frac{1}{4\pi\varepsilon_0} = 8.988\times10^{9} N\cdot m^2\cdot C^{-2}$	
	$\frac{e^2}{4\pi\varepsilon_0} = 1.44 eV\cdot nm$	