PHYS 2380 Quantum Physics 1

Lecture 4 – Boltzmann Distribution and Rayleigh-Jeans and Planck Theories for Blackbody radiation

Outline

- How do we develop a theory to explain our observations for thermal radiation?
- The Boltzmann distribution for energy
- Modes of oscillation in a cavity
- Rayleigh-Jeans theory and its failure
- Planck theory for black body radiation

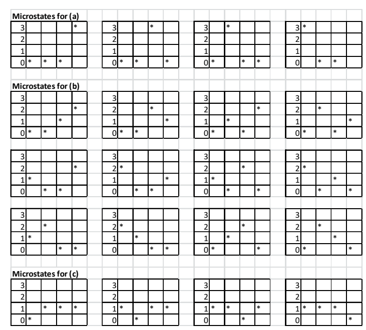
The Boltzmann Distribution

- Independently derived by both Maxwell and Boltzmann
- Used to construct the Kinetic Theory of Gases
- Experimentally verified in many applications
- For a system containing a number of objects:
- in thermal equilibrium, the system has a well defined total energy
- The objects can exchange energy with each other
- Individual energies vary randomly, but the average energy per object is well defined
- Detailed mathematical proof is available; we will use a "plausibility argument" to get to the result here.

Plausibility argument

- Consider a small system of 4 objects
- Set the total energy of the system as 3ΔE, where ΔE is some amount of energy
- Restrict the energy of each object to values of 0, 1ΔE, 2ΔE, 3ΔE,…
- What are the various ways that the total energy can be distributed among the population?

Microstates for the system



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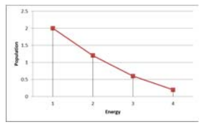
		Energy per object				Num. of	Prob. Of
State	0	ΔE	2ΔE	3ΔE	4ΔE	Config.	Occur.
a	***			*		4	"4/20"
b	**	*	*			12	"12/20"
c	*	***				4	"4/20"
			Total num. of cor			20	

Average populations in each bin

- Average number of objects in each bin = Σ(# of particles in a bin) x (probability of finding the state)

$$E = 0\Delta E\ bin: \left(3\times {}^{4}\!/_{20}\right)+\left(2\times {}^{12}\!/_{20}\right)+\left(1\times {}^{4}\!/_{20}\right)= {}^{40}\!/_{20}$$
$$E = 1\Delta E\ bin: \left(0\times {}^{4}\!/_{20}\right)+\left(1\times {}^{12}\!/_{20}\right)+\left(3\times {}^{4}\!/_{20}\right)= {}^{24}\!/_{20}$$
$$E = 2\Delta E\ bin: \left(0\times {}^{4}\!/_{20}\right)+\left(1\times {}^{12}\!/_{20}\right)+\left(0\times {}^{4}\!/_{20}\right)= {}^{12}\!/_{20}$$
$$E = 3\Delta E\ bin: \left(1\times {}^{4}\!/_{20}\right)+\left(0\times {}^{12}\!/_{20}\right)+\left(0\times {}^{4}\!/_{20}\right)= {}^{4}\!/_{20}$$
$$E = 4\Delta E\ bin: 0$$



- Verify: $\sum_{all\ E}(\bar{n}_i) = \left({}^{40}\!/_{20}+{}^{24}\!/_{20}+{}^{12}\!/_{20}+{}^{4}\!/_{20}\right)=4$
  - The total number of particles in our system!

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In the limit…

- Increase the number of particles
- Make the energy bins finer
- We get the Boltzmann distribution: $n(\varepsilon)d\varepsilon = Ae^{-\varepsilon/\varepsilon_0}d\varepsilon$
- Probability of finding a particle with energy between ε and ε+dε: $P(\varepsilon)d\varepsilon = \frac{n(\varepsilon)d\varepsilon}{N}$
  - Where N is the total number of objects in the system
- P(ε) is called the probability density
  - C is a constant to be determined

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Normalization

- We know the total number of particles in our system
- Can use this to determine C:

$$\int_0^\infty P(\varepsilon)d\varepsilon = \int_0^\infty Ce^{-\varepsilon/\varepsilon_0}d\varepsilon = 1$$
$$P(\varepsilon)d\varepsilon = \frac{A}{N}e^{-\varepsilon/\varepsilon_0}d\varepsilon = Ce^{-\varepsilon/\varepsilon_0}d\varepsilon$$
$$= C(-\varepsilon_0)e^{-\varepsilon/\varepsilon_0}\Big]_0^\infty = -C\varepsilon_0(0-1)=1$$
$$\therefore C\varepsilon_0 = 1 \Rightarrow C = {}^{1}\!/_{\varepsilon_0}$$

In summary:

$$P(\varepsilon)d\varepsilon = \frac{1}{\varepsilon_0}e^{-\varepsilon/\varepsilon_0}d\varepsilon$$

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Average energy

- We can use our distribution to evaluate the average energy per particle in our system: $\bar{\varepsilon} = \langle\varepsilon\rangle = \int_0^\infty \varepsilon P(\varepsilon)d\varepsilon$
  - NB that we use the normalized form for P(e) here.
- After some calculus and algebra we obtain the result $\bar{\varepsilon} = \langle\varepsilon\rangle = \varepsilon_0$
- From the kinetic theory of gases we get $\varepsilon_0 = kT\ J$, $where\ k = 1.38\times10^{-23}\ J/K$

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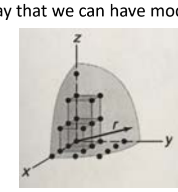
Modes of 0scillation in a cavity

- Our blackbody cavity is filled with electromagnetic waves
- How are the allowed waves distributed as a function of wavelength (or frequency)?
- If we look at the standing waves inside a cube of side L, the solutions to the wave equations say that we can have modes such that:

$$n_x^2 + n_y^2 + n_z^2 = {}^{4L^2}\!/_{\lambda^2}$$

- Where the "n's" are all integers



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Modes of Oscillation (contd.)

- The modes fill one quadrant of a sphere in n-space.
- The allowed lattice points have a density of 1
- The number of allowed modes up to a particular value of 1/λ is given by the volume enclosed by:
- We want the incremental number of modes when we increase λ to λ+dλ:
  - There is a factor of 2 here because we have two polarizations for light
- To summarize the distribution of modes between values of λ to λ+dλ, **per unit volume** will be:

$$r = {}^{2L}\!/_{\lambda}\ \ so:$$
$$n = \left(\frac{4}{3}\pi r^3\right)\times\frac{1}{8} = \frac{4}{3}\pi\frac{8L^3}{\lambda^3}\times\frac{1}{8} = \frac{4\pi L^3}{3\lambda^3}$$
$$dn = \frac{4\pi L^3}{\lambda^4}d\lambda$$
$$n(\lambda)d\lambda = \frac{8\pi}{\lambda^4}d\lambda$$

converting to frequency:

$$n(\nu) = \frac{8\pi\nu^2}{c^3}d\nu$$

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Putting it all together…

- Rayleigh-Jeans theory:
- From kinetic theory of gases: average energy per entity: *kT*
- Density of modes between λ and $\lambda+\Delta\lambda$:

$$n(\lambda)d\lambda = \frac{8\pi}{\lambda^4}d\lambda$$

- Energy density inside cavity:

$$u(\lambda)d\lambda = \langle\varepsilon\rangle\left(\frac{8\pi}{\lambda^4}d\lambda\right) = kT\left(\frac{8\pi}{\lambda^4}d\lambda\right) \qquad u(\nu)d\nu = kT\left(\frac{8\pi\nu^2}{c^3}d\nu\right)$$

Rayleigh-Jeans result

– The energy density moves with a speed of *c*
– Only half the radiation moves towards the opening at any time
– The radiation approaches from different angles so there is another geometric factor of ½.

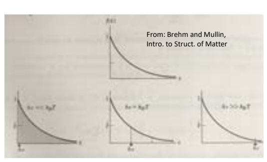
$$R(\lambda)d\lambda = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)cu(\lambda)d\lambda$$

$$= \frac{8\pi ckT}{\lambda^4}d\lambda$$

– R(λ) diverges for short λ, infinite energy density:
– **Ultraviolet catastrophe!** Because the average energy per mode is constant
– **Need a better result.**

Planck's solution

- Planck suggested not to use: $\langle\varepsilon\rangle = \frac{1}{\varepsilon_0}\int_0^\infty \varepsilon e^{-\varepsilon/\varepsilon_0}d\varepsilon = kT$
- The modes should not all have the same average energy.
- Modes should only have distinct values for energies: $E = h\nu, 2h\nu, 3h\nu, \ldots$
- Use a sum over these different energy values.
- This de-emphasizes the contributions of the shorter wavelengths
- The value of ***h*** to be determined from the best fit to the spectrum



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$$\langle\varepsilon\rangle = \frac{\sum_n \varepsilon_n f(\varepsilon_n)}{\sum_n f(\varepsilon_n)} = \frac{\sum_n nh\nu e^{-nh\nu/kT}}{\sum_n e^{-nh\nu/kT}} = \frac{h\nu}{e^{-nh\nu/kT}-1} = \frac{hc}{\lambda\left(e^{hc/\lambda kT}-1\right)}$$

- The value of ***h*** to be determined from the best fit to the spectrum
- This de-emphasizes the contributions of the shorter wavelengths
- ***h= 6.626 x 10^{-34} J-s***

Planck's result

- This changes the expression to:

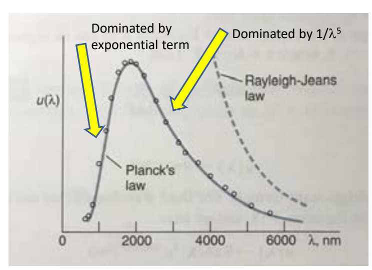
$$u(\lambda)d\lambda = \frac{8\pi hc}{\lambda^5}\frac{d\lambda}{e^{hc/\lambda kT}-1} \qquad u(\nu)d\nu = \frac{8\pi h\nu^3}{c^3}\frac{d\nu}{e^{h\nu/kT}-1}$$

- Multiplying by (c/4) to obtain the radiance:

$$R(\lambda)d\lambda = \frac{2\pi hc^2}{\lambda^5}\frac{d\lambda}{e^{hc/\lambda kT}-1}\ W/m^2 \qquad R(\nu)d\nu = \frac{2\pi h\nu^3}{c^2}\frac{d\nu}{e^{h\nu/kT}-1}\ W/m^2$$

Black body radiation curve

- From: p.127 of Llewelyn and Tippler



Stefan-Boltzmann Law from Planck

- Stefan Boltzman Law states that:

$$R_{tot} = \int_0^\infty R(\lambda) d\lambda = \sigma T^4$$

$$= \int_0^\infty \frac{2\pi hc^2}{\lambda^5} \frac{1}{e^{hc/\lambda kT} - 1} d\lambda = 2\pi hc^2 \left(\frac{kT}{hc}\right)^5 \int_0^\infty \left(\frac{hc}{\lambda kT}\right)^5 \frac{1}{e^{hc/\lambda kT} - 1} d\lambda$$

- Changing variable to: $x = \frac{hc}{\lambda kT}$

$$dx = -\left(\frac{hc}{kT}\right)\left(\frac{-1}{\lambda^2}\right) d\lambda = -\left(\frac{kT}{hc}\right)\left(\frac{hc}{\lambda kT}\right)^2 d\lambda = -\left(\frac{kT}{hc}\right) x^2 d\lambda$$

$$\therefore d\lambda = -\left(\frac{hc}{kT}\right)\left(1/x^2\right) dx$$

SBL from Planck contd.

- The integral transforms into:

$$R_{tot} = -\left(2\pi hc^2\right)\left(\frac{kT}{hc}\right)^4 \int_{\lambda=0}^{\lambda=\infty} \frac{x^3}{e^x - 1} dx = -\left(2\pi hc^2\right)\left(\frac{kT}{hc}\right)^4 \int_{x=\infty}^{x=0} \frac{x^3}{e^x - 1} dx$$

$$= \left(2\pi hc^2\right)\left(\frac{kT}{hc}\right)^4 \int_{x=0}^{x=\infty} \frac{x^3}{e^x - 1} dx = \left(2\pi hc^2\right)\left(\frac{kT}{hc}\right)^4 \left(\frac{\pi^4}{15}\right)$$

$$= \left(\frac{2\pi^5 k^4}{15 h^3 c^2}\right) T^4 = \sigma T^4$$

h	6.62607E-34
k	1.38065E-23
c	299792458
σ	5.67037E-08

- Success!

Summary: Blackbody radiation

- Wien displacement law: $\lambda_m T = 2.898 \times 10^{-3}\ m\,K$
- Stefan-Boltzmann law: $R_{tot} = \sigma T^4\ W/m^2$
- Used thermodynamic concepts, standing waves, and a Boltzmann distribution modified by Planck to arrive at a comprehensive theory for the radiation.
- There was a glimpse of quantization: $E = nh\nu \text{ where } n = 1, 2, 3...$
- Energy density inside the cavity:

$$u(\lambda) d\lambda = \frac{8\pi hc}{\lambda^5} \frac{d\lambda}{e^{hc/\lambda kT} - 1}\ J/m^3 \qquad u(\nu) d\nu = \frac{8\pi h\nu^3}{c^3} \frac{d\nu}{e^{h\nu/kT} - 1}\ J/m^3$$

- Radiance:

$$R(\lambda) d\lambda = \frac{2\pi hc^2}{\lambda^5} \frac{d\lambda}{e^{hc/\lambda kT} - 1}\ W/m^2 \qquad R(\nu) d\nu = \frac{2\pi h\nu^3}{c^2} \frac{d\nu}{e^{h\nu/kT} - 1}\ W/m^2$$