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RESEARCH ARTICLES

A Weaker Smoothness Criterion for the Inverse Function Theorem, the Intermediate Value Theorem, and the Mean Value Theorem in a non-Archimedean Setting

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Abstract—We introduce a class of so-called very weakly locally uniformly differentiable (VWLUD) functions at a point of a general non-Archimedean ordered field extension of the real numbers, $\mathcal{N}$, which is real closed and Cauchy complete in the topology induced by the order, and whose Hahn group is Archimedean. This new class of functions is defined by a significantly weaker criterion than that of the class of weakly locally uniformly differentiable (WLUD) functions studied in [1], which is nonetheless sufficient for a slight variation of the inverse function theorem and intermediate value theorem. Similarly, a weaker second order criterion is derived from the previously studied WLUD^2 condition for twice-differentiable functions. We show that VWLUD^2 functions at a point of $\mathcal{N}$ satisfy the mean value theorem in an interval around that point.

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1. INTRODUCTION

Let $\mathcal{N}$ be a non-Archimedean ordered field extension of $\mathbb{R}$ that is real closed and complete in the order topology and whose Hahn group $S_{\mathcal{N}}$ is Archimedean, i.e. (isomorphic to) a subgroup of $\mathbb{R}$. Recall that $S_{\mathcal{N}}$ is the set of equivalence classes under the relation $\sim$ defined on $\mathcal{N}^* := \mathcal{N} \setminus \{0\}$ as follows: For $x, y \in \mathcal{N}^*$, we say that x is of the same order as y and write $x \sim y$ if there exist $n, m \in \mathbb{N}$ such that $n|x| > |y|$ and $m|y| > |x|$, where $|\cdot|$ denotes the ordinary absolute value on $\mathcal{N}$: $|x| = \max\{x, -x\}$. $S_{\mathcal{N}}$ is naturally endowed with an addition via $[x] + [y] = [x \cdot y]$ and an order via $[x] \leq [y]$ if $[x] = [y]$ or $|y| \ll |x|$ (which means $n|y| < |x|$ for all $n \in \mathbb{N}$), both of which are readily checked to be well-defined. It follows that $(S_{\mathcal{N}}, +, <)$ is an ordered group, often referred to as the Hahn group or skeleton group, whose neutral element is $[1]$, the class of 1.

The theorem of Hahn [3] provides a complete classification of non-Archimedean ordered field extensions of $\mathbb{R}$ in terms of their skeleton groups. In fact, invoking the axiom of choice, it is shown that the elements of our field $\mathcal{N}$ can be written as (generalized) formal power series (also called Hahn series) over its skeleton group $S_{\mathcal{N}}$ with real coefficients, and the set of appearing exponents forms a well-ordered subset of $S_{\mathcal{N}}$. That is, for all $x \in \mathcal{N}$, we have that $x = \sum_{q \in S_{\mathcal{N}}} a_q d^q$; with $a_q \in \mathbb{R}$ for all q, d a positive infinitely small element of $\mathcal{N}$, and the support of x, given by $\operatorname{supp}(x) := \{q \in S_{\mathcal{N}} : a_q \neq 0\}$, forming a well-ordered subset of $S_{\mathcal{N}}$.

We define for $x \neq 0$ in $\mathcal{N}$, $\lambda(x) = \min(\operatorname{supp}(x))$, which exists since $\operatorname{supp}(x)$ is well-ordered. Moreover, we set $\lambda(0) = \infty$. Given a nonzero $x = \sum_{q \in \operatorname{supp}(x)} a_q d^q$, then $x > 0$ if and only if $a_{\lambda(x)} > 0$.

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The smallest such field $\mathcal{N}$ is the Levi-Civita field $\mathcal{R}$, first introduced in [5, 6]. In this case $S_{\mathcal{R}} = \mathbb{Q}$, and for any element $x \in \mathcal{R}$, $\operatorname{supp}(x)$ is a left-finite subset of $\mathbb{Q}$, i.e. below any rational bound r there are only finitely many exponents in the Hahn representation of x. The Levi-Civita field $\mathcal{R}$ is of particular interest because of its practical usefulness. Since the supports of the elements of $\mathcal{R}$ are left-finite, it is possible to represent these numbers on a computer. Having infinitely small numbers allows for many computational applications; one such application is the computation of derivatives of real functions representable on a computer [11, 12], where both the accuracy of formula manipulators and the speed of classical numerical methods are achieved. For a review of the Levi-Civita field $\mathcal{R}$, see [10] and the references therein.

In the wider context of valuation theory, it is interesting to note that the topology induced by the order on $\mathcal{N}$ is the same as the valuation topology τ_v introduced via the non-Archimedean (ultrametric) valuation $|\cdot|_v : \mathcal{N} \to \mathbb{R}$, given by

$$|x|_v = \begin{cases} \exp\left(-\lambda(x)\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

It follows therefore that the field $\mathcal{N}$ is just a special case of the class of fields discussed in [9]. For a general overview of the algebraic properties of formal power series fields, we refer to the comprehensive overview by Ribenboim [8], and for an overview of the related valuation theory, to the book by Krull [4]. A thorough and complete treatment of ordered structures can also be found in [7]. A more comprehensive survey of all non-Archimedean fields can be found in [2].

Because of the total disconnectedness of the field $\mathcal{N}$ in the order topology, the standard theorems of real calculus like the intermediate value theorem, the inverse function theorem, the mean value theorem, the implicit function theorem and Taylor's theorem require stronger smoothness criteria of the functions involved in order for the theorems to hold.

In [15] we studied the properties of locally uniformly differentiable (LUD) functions at a point or on an open subset of $\mathcal{N}$. In particular, we showed that this class of functions is closed under addition, multiplication and composition of functions. Then we stated and proved local versions of the inverse function theorem and the intermediate value theorem for $\mathcal{N}$-valued LUD functions in an open neighborhood of a point $x_0 \in \mathcal{N}$. Then, in [13], we generalized the definition of local uniform differentiability to any order. Then we studied the properties of n-times locally uniformly differentiable (LUD^n) functions and we formulated and proved a local mean value theorem for $\mathcal{N}$-valued functions that are LUD^2 in a neighborhood of a point of $\mathcal{N}$.

In [1], we introduced a new smoothness criterion which we called weakly local uniform differentiability (WLUD) which is strictly weaker than local uniform differentiability and strictly stronger than continuous differentiability (C^1). We studied the properties of $\mathcal{N}$-valued WLUD and WLUD^n functions and we showed that this weaker criterion is sufficient to get all the nice calculus results obtained in [13, 15].

In this paper, we weaken the smoothness criterion further by introducing and studying the so-called very weakly locally uniformly differentiable (VWLUD) functions. As the name implies, this class of functions strictly contains all WLUD functions, but is still strictly contained in the class of continuous functions. Since VWLUD functions may only be differentiable at a single point as we will see later in the paper, this weakening of the WLUD criterion is not insignificant. Even so, we will show that the new VWLUD criterion at just one point $x_0 \in \mathcal{N}$ is sufficient for the local inverse function theorem and intermediate value theorem, and a related VWLUD^2 criterion at a point $x_0 \in \mathcal{N}$ will suffice for the mean value theorem to hold in an interval around x_0.

2. DEFINITIONS

Throughout this paper, given $x_0 \in \mathcal{N}$ and $\delta > 0$ in $\mathcal{N}$, the open interval $\{x \in \mathcal{N} : |x - x_0| < \delta\}$, of length 2δ centered at x_0, will be denoted by $B(x_0, \delta)$ and it will sometimes be referred to as the open "bal" of radius δ centered at x_0.

Definition 2.1 (WLUD). *Let A be an open subset of $\mathcal{N}$ and let $f : A \to \mathcal{N}$. Given an element $x_0 \in A$, we say that f is weakly locally uniformly differentiable (WLUD) at x_0 if there is an open neighbourhood Ω of x_0 in A in which f is differentiable, and if for every $\epsilon > 0$ in $\mathcal{N}$ there exists $\delta > 0$ in $\mathcal{N}$ such that $B(x_0, \delta) \subset \Omega$ and*

$$x, y \in B(x_0, \delta) \implies |f(y) - f(x) - f'(x)(y - x)| \leq \epsilon |y - x|.$$

Definition 2.2 (VWLUD). *Let $A \subseteq \mathcal{N}$ be open and let $f : A \to \mathcal{N}$. Given $x_0 \in A$, we say that f is very weakly locally uniformly differentiable (VWLUD) at x_0 if there exists an open neighbourhood Ω of x_0 in which f is continuous, if f is differentiable at x_0, and if for every $\epsilon > 0$ in $\mathcal{N}$ there exists $\delta > 0$ in $\mathcal{N}$ such that $B(x_0, \delta) \subset \Omega$ and*

$$x, y \in B(x_0, \delta) \implies |f(y) - f(x) - f'(x_0)(y - x)| \leq \epsilon |y - x|.$$

As WLUD functions are C^1 [1], it is easily seen that if f is WLUD at x_0 then it is VWLUD at x_0. Conversely, if f is VWLUD at x_0 and if there exists a $\delta > 0$ in $\mathcal{N}$ such that f is differentiable at all $x \in B(x_0, \delta)$, then it can be shown that f is WLUD at x_0 (Proposition 11, [1]).

As well as to show that the VWLUD condition is strictly weaker than the WLUD condition, it is useful to have an example of a function which is VWLUD at some point x_0, but not WLUD at x_0. From the above observation, this function must then not be differentiable in any neighbourhood of x_0. With this in mind, we will construct such a function with $x_0 = 0$ so that given any neighbourhood U of 0, the function will be differentiable everywhere on U except at a countable number of points.

Example 2.3. *Define $g_1 : [d, 1) \to \mathcal{N}$ by*

$$g_1(x) = (1 - d)x.$$

and for all $n \in \#N \setminus \{1\}$, define $g_n : [d^n, d^{n-1})$ by

$$g_n(x) = (1 - d^n)x - \sum_{j=2}^{2n-1} (-d)^j.$$

Given $x \in (0, 1)$, we remark that there exists a unique $m \in \#N$ with the property that $d^m \leq x < d^{m-1}$ and if $d^n \leq x$, then $n \geq m$. That is, m is the smallest positive integer such that $d^m \leq x < d^{m-1}$. For $x \in (-1, 0)$, we define the m corresponding to x to be exactly the m corresponding to $|x| = -x$. With this in mind, let $f : (-1, 1) \to \mathcal{N}$ be given by

$$f(x) = \begin{cases} g_m(x) + \frac{d^2}{1+d}, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -g_m(-x) - \frac{d^2}{1+d}, & \text{if } x < 0. \end{cases}$$

It can be checked that f is an odd function and continuous on $(-1, 1)$.

Due to its piecewise characterization, it is clear that f is not differentiable at $\pm d^n$ for every $n \in \#N$, but that $f'(x) = 1 - d^m$ for every $x \in (-1, 1) \setminus \{0\}$ with $x \neq \pm d^n$ for any $n \in \#N$. We now show that f is differentiable at $x = 0$ with $f'(0) = 1$.

Given $\epsilon > 0$ in $\mathcal{N}$, choose $n \in \#N$ such that $d^n \ll \epsilon$ and let $\delta = d^n$. Then, given $x \in (0, \delta)$, we have that

$$\begin{aligned} \left| \frac{f(x) - f(0)}{x - 0} - 1 \right| &= \left| \frac{f(x)}{x} - 1 \right| \\ &= \left| \frac{(1 - d^m)x - \sum_{j=2}^{2m-1} (-d)^j + \frac{d^2}{1+d}}{x} - 1 \right| \\ &= \left| \frac{(1 - d^m)x - x}{x} + \frac{\frac{d^2}{1+d} - \sum_{j=2}^{2m-1} (-d)^j}{x} \right| \\ &\leq |(1 - d^m) - 1| + \frac{1}{|x|} \left| \frac{d^2}{1+d} - \sum_{j=2}^{2m-1} (-d)^j \right| \\ &= d^m + \frac{1}{x} \left| \sum_{j=2}^{\infty} (-d)^j - \sum_{j=2}^{2m-1} (-d)^j \right| \end{aligned}$$

$$
\begin{aligned}
&< d^m + \frac{1}{d^m}\sum_{j=2m}^{\infty}(-d)^j \\
&= d^m + \left(d^m - d^{m+1} + d^{m+2} - \cdots\right) \\
&\le d^m + d^m \\
&= 2d^m \ll 2d^n \ll \epsilon.
\end{aligned}
$$

As f is odd, the same argument can be used for $x \in (-\delta, 0)$. Thus,

$$
f'(x) = \begin{cases} 1 - d^m, & \text{if } x \neq 0 \text{ and } x \neq d^{\pm n} \text{ for any } n \in \#N \\ 1, & \text{if } x = 0. \end{cases}
$$

Next, we show that f is VWLUD at $x = 0$. Given $\epsilon > 0$ in $\mathcal{N}$, choose $n \in \#N$ such that $d^n \ll \epsilon$ and let $\delta = d^n$. Let $x, y \in B(0, \delta)$. We let l correspond to y as m corresponds to x, and consider a few cases (and subcases):

Case 1. $x, y > 0$. Then we have that

$$
\begin{aligned}
\left|f(y) - f(x) - f'(0)(y - x)\right| &= \left|(1 - d^l)y - \sum_{j=2}^{2l-1}(-d)^j - (1 - d^m)x + \sum_{j=2}^{2m-1}(-d)^j - (y - x)\right| \\
&= \left|d^m x - d^l y + \sum_{j=2}^{2m-1}(-d)^j - \sum_{j=2}^{2l-1}(-d)^j\right| \\
&\le |d^m x - d^l y| + \left|\sum_{j=2}^{2m-1}(-d)^j - \sum_{j=2}^{2l-1}(-d)^j\right|.
\end{aligned}
$$

If $l = m$, then

$$
\left|f(y) - f(x) - f'(0)(y - x)\right| = |d^m x - d^m y| = d^m|y - x| \ll d^n|y - x| \ll \epsilon|y - x|.
$$

If $l < m$, then

$$
\left|f(y) - f(x) - f'(0)(y - x)\right| = |d^m x - d^l y| + \sum_{j=2l}^{2m-1}(-d)^j.
$$

We remark that

$$
\lambda(\epsilon|y - x|) = \lambda(\epsilon) + \lambda|y - x| < n + \lambda(y) < l + \lambda(y) \le l + l = 2l.
$$

So, as $\lambda\left(\sum_{j=2l}^{2m-1}(-d)^j\right) = 2l > \lambda(\epsilon|y - x|)$ and

$$
\lambda(|d^m x - d^l y|) = \lambda(d^l|y - d^{m-l}x|) = l + \lambda(y) > \lambda(\epsilon|y - x|),
$$

it follows that

$$
|d^m x - d^l y| \ll \epsilon|y - x| \quad \text{and} \quad \sum_{j=2l}^{2m-1}(-d)^j \ll \epsilon|y - x|.
$$

Thus,

$$
|f(y) - f(x) - f'(0)(y - x)| \ll \epsilon|y - x|.
$$

If $l > m$, the argument is essentially the same as in the case with $l < m$.

Case 2. $x, y < 0$. Then we can use the same arguments used in Case 1 above since f is an odd function.

Case 3. x and y have opposite signs. Then, without loss of generality, we may assume $x < 0 < y$. Thus,

$$\begin{aligned}
&|f(y) - f(x) - f'(0)(y - x)| \\
&= \left|(1 - d^l)y - \sum_{j=2}^{2l-1}(-d)^j + \frac{d^2}{1+d} + (1 - d^m)(-x) - \sum_{j=2}^{2m-1}(-d)^j + \frac{d^2}{1+d} - (y - x)\right| \\
&\leq \left|d^m x - d^l y\right| + \left|\frac{d^2}{1+d} - \sum_{j=2}^{2l-1}(-d)^j\right| + \left|\frac{d^2}{1+d} - \sum_{j=2}^{2m-1}(-d)^j\right| \\
&= |d^m x - d^l y| + \left|\sum_{j=2}^{\infty}(-d)^j - \sum_{j=2}^{2l-1}(-d)^j\right| + \left|\sum_{j=2}^{\infty}(-d)^j - \sum_{j=2}^{2m-1}(-d)^j\right| \\
&= |d^m x - d^l y| + \sum_{j=2l}^{\infty}(-d)^j + \sum_{j=2m}^{\infty}(-d)^j.
\end{aligned}$$

In the exact same way as in Case 1, we can analyze the λ value of each term above to show that every term is $\ll \epsilon|y - x|$.

Thus, f is VWLUD at 0. However, f is not WLUD at 0, as there is no neighbourhood of 0 in which f is differentiable.

3. PROPERTIES OF VWLUD FUNCTIONS

As noted earlier in the paper, if a function f is VWLUD and C^1 at x_0 then f is necessarily WLUD at x_0. As the theory of WLUD functions has already been investigated, we are interested in studying functions which are VWLUD but not C^1. So, we define a new class of functions which contains the class of C^1 functions.

Definition 3.1 (WC ^{1}). *Let $A \subseteq \mathcal{N}$ be open and let $f : A \to \mathcal{N}$. Given $x_0 \in A$, we say that f is weakly C^1 (WC^1) at x_0 if f is differentiable at x_0 and if for every $\epsilon > 0$ in $\mathcal{N}$ there exists $\delta > 0$ in $\mathcal{N}$ such that*

$$(x \in B(x_0, \delta) \wedge f'(x) \text{ exists}) \implies |f'(x) - f'(x_0)| < \epsilon.$$

Remark 3.2. *It is readily seen that C^1 functions at a point x_0 are WC^1 at x_0.*

Proposition 3.3. *Let $A \subseteq \mathcal{N}$ be open and let $f : A \to \mathcal{N}$ be VWLUD at $x_0 \in A$. Then f is WC^1 at x_0.*

Proof. Let $\epsilon > 0$ in $\mathcal{N}$ be given. Then there exists $\delta > 0$ in $\mathcal{N}$ such that $B(x_0, \delta) \subset A$ and, if $s, t \in B(x_0, \delta)$ are distinct, then

$$|f(s) - f(t) - f'(x_0)(s - t)| < \frac{\epsilon}{2}|s - t|$$

or, equivalently,

$$\left|\frac{f(s) - f(t)}{s - t} - f'(x_0)\right| < \frac{\epsilon}{2}.$$

Given $x \in B(x_0, \delta)$ such that $f'(x)$ exists, then we can find a $y \neq x$ in $B(x_0, \delta)$ such that

$$\left|\frac{f(y) - f(x)}{y - x} - f'(x)\right| < \frac{\epsilon}{2}.$$

Thus,

$$
\begin{aligned}
|f'(x) - f'(x_0)| &\le \left| f'(x) - \frac{f(y) - f(x)}{y - x} \right| + \left| \frac{f(y) - f(x)}{y - x} - f'(x_0) \right| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}
$$

□

With the following proposition, it is shown that though a function f which is VWLUD at x_0 may not be WLUD at this point, we may still find a small ball around x_0 in which we obtain the WLUD inequality at only the points at which f is differentiable. This demonstrates the fact that differentiability is the factor which separates the VWLUD criterion from that of the WLUD.

Proposition 3.4. *Let $A \subseteq \mathcal{N}$ be open and let $f : A \to \mathcal{N}$ be VWLUD at $x_0 \in A$. Then, for every $\epsilon > 0$ in $\mathcal{N}$, there exists $\delta > 0$ in $\mathcal{N}$ such that $B(x_0, \delta) \subset A$ and*

$$
(x, y \in B(x_0, \delta) \wedge f'(x) \text{ exists}) \implies |f(y) - f(x) - f'(x)(y - x)| \le \epsilon |y - x|.
$$

Proof. Let $\epsilon > 0$ in $\mathcal{N}$ be given. Since A is open and since f is both VWLUD and WC^1 at x_0, there exists $\delta > 0$ such that $B(x_0, \delta) \subset A$ and

$$
|f(y) - f(x) - f'(x_0)(y - x)| \le \frac{\epsilon}{2} |y - x|
$$

for all $x, y \in B(x_0, \delta)$, and

$$
|f'(x) - f'(x_0)| \le \frac{\epsilon}{2}
$$

for all $x \in B(x_0, \delta)$ such that $f'(x)$ exists. Thus, given $x, y \in B(x_0, \delta)$ such that f is differentiable at x, we have that

$$
\begin{aligned}
|f(y) - f(x) - f'(x)(y - x)| &\le |f(y) - f(x) - f'(x_0)(y - x)| \\
&\quad + |f'(x_0)(y - x) - f'(x)(y - x)| \\
&\le \frac{\epsilon}{2} |y - x| + \frac{\epsilon}{2} |y - x| = \epsilon |y - x|.
\end{aligned}
$$

□

Like WLUD functions, the class of VWLUD functions has the important quality of being closed under addition, multiplication and composition.

Proposition 3.5. *Let $A \subseteq \mathcal{N}$ be open, let $f, g : A \to \mathcal{N}$ be VWLUD at $x_0 \in A$, and let $\alpha \in \mathcal{N}$ be given. Then $(f + \alpha g)$ is VWLUD at x_0.*

Proof. If $\alpha = 0$, then the result holds trivially, so assume $\alpha \neq 0$. Since f and g are both VWLUD at x_0, there exist open neighbourhoods Ω_f and Ω_g of x_0 in A such that f is continuous on Ω_f and g is continuous on Ω_g. Let $\Omega = \Omega_f \cap \Omega_g$. Then Ω is an open neighbourhood of x_0 in A in which $f + \alpha g$ is continuous. Now let $\epsilon > 0$ in $\mathcal{N}$ be given. As f is VWLUD at x_0, there exists a $\delta_f > 0$ in $\mathcal{N}$ such that $B(x_0, \delta_f) \subset \Omega$ and, for every $x, y \in B(x_0, \delta_f)$, we have that

$$
|f(y) - f(x) - f'(x_0)(y - x)| \le \frac{\epsilon}{2} |y - x|.
$$

Likewise, there is a $\delta_g > 0$ in $\mathcal{N}$ such that $B(x_0, \delta_g) \subset \Omega$ and, for every $x, y \in B(x_0, \delta_g)$, we have that

$$
|g(y) - g(x) - g'(x_0)(y - x)| \le \frac{\epsilon}{2|\alpha|} |y - x|.
$$

Let $\delta = \min\{\delta_f, \delta_g\}$. Then, $B(x_0, \delta) \subset \Omega$ and, for every $x, y \in B(x_0, \delta)$, we have that

$$
|(f + \alpha g)(y) - (f + \alpha g)(x) - (f' + \alpha g')(x_0)(y - x)|
$$

$$
\begin{aligned}
&\le |f(y)-f(x)-f'(x_0)(y-x)| + |\alpha||g(y)-g(x)-g'(x_0)(y-x)| \\
&\le \frac{\epsilon}{2}|y-x| + |\alpha|\frac{\epsilon}{2|\alpha|}|y-x| \\
&= \epsilon|y-x|.
\end{aligned}
$$

□

Proposition 3.6. *Let $A \subseteq \mathcal{N}$ be open and let $f, g : A \to \mathcal{N}$ be VWLUD at $x_0 \in A$. Then fg is VWLUD at x_0.*

Proof. As in the proof of Proposition 3.5 above, since f and g are VWLUD at x_0, there exists an open neighbourhood Ω of x_0 in A such that fg is continuous on Ω. Let $\epsilon > 0$ in $\mathcal{N}$ be given. Then, as f and g are VWLUD at x_0, there exist $\delta_f > 0$ and $\delta_g > 0$ in $\mathcal{N}$ such that $B(x_0, \delta_f) \subset \Omega$, $B(x_0, \delta_g) \subset \Omega$,

$$
|f(y)-f(x)-f'(x_0)(y-x)| \le \frac{\epsilon}{4(|g(x_0)|+1)}|y-x|
$$

for all $x, y \in B(x_0, \delta_f)$, and

$$
|g(y)-g(x)-g'(x_0)(y-x)| \le \frac{\epsilon}{4(|f(x_0)|+1)}|y-x|
$$

for all $x, y \in B(x_0, \delta_g)$.

By continuity of f and g at x_0, there exist $\delta_1 > 0$ and $\delta_2 > 0$ in $\mathcal{N}$ such that $B(x_0, \delta_1) \subset \Omega$, $B(x_0, \delta_2) \subset \Omega$,

$$
|f(x)-f(x_0)| < \min\left\{\frac{\epsilon}{4(|g'(x_0)|+1)}, 1\right\}
$$

for all $x \in B(x_0, \delta_1)$, and

$$
|g(x)-g(x_0)| < \min\left\{\frac{\epsilon}{4(|f'(x_0)|+1)}, 1\right\}
$$

for all $x \in B(x_0, \delta_2)$.

Let $\delta = \min\{\delta_f, \delta_g, \delta_1, \delta_2\}$. Then $B(x_0, \delta) \subset \Omega$ and, for all $x, y \in B(x_0, \delta)$, we have that

$$
\begin{aligned}
&\left|f(y)g(y)-f(x)g(x)-\big(f'(x_0)g(x_0)+f(x_0)g'(x_0)\big)(y-x)\right| \\
&\le \left|f(y)g(y)-f(x)g(y)-f'(x_0)g(y)(y-x)\right| + \left|f(x)g(y)-f(x)g(x)-f(x)g'(x_0)(y-x)\right| \\
&\quad + \left|f'(x_0)g(y)-f'(x_0)g(x_0)\right||y-x| + \left|f(x)g'(x_0)-f(x_0)g'(x_0)\right||y-x| \\
&= |g(y)|\left|f(y)-f(x)-f'(x_0)(y-x)\right| + |f(x)|\left|g(y)-g(x)-g'(x_0)(y-x)\right| \\
&\quad + |f'(x_0)||g(y)-g(x_0)||y-x| + |g'(x_0)||f(x)-f(x_0)||y-x| \\
&\le \frac{|g(y)|}{4(|g(x_0)|+1)}\epsilon|y-x| + \frac{|f(x)|}{4(|f(x_0)|+1)}\epsilon|y-x| \\
&\quad + \frac{|f'(x_0)|}{4(|f'(x_0)|+1)}\epsilon|y-x| + \frac{|g'(x_0)|}{4(|g'(x_0)|+1)}\epsilon|y-x| \\
&\le \epsilon|y-x|.
\end{aligned}
$$

□

Remark 3.7. *It follows from Proposition 3.5 and Proposition 3.6 that VWLUD functions at a point $x_0 \in \mathcal{N}$ form an $\mathcal{N}$-algebra.*

Remark 3.8. *As polynomials are WLUD at every $t \in \mathcal{N}$ [1], they are also VWLUD at every $t \in \mathcal{N}$. Alternatively, this can be shown by demonstrating that $f(x) = x$ is VWLUD at every $t \in \mathcal{N}$ and applying the previous two propositions.*

Proposition 3.9. *Let $A, B \subseteq \mathcal{N}$ be open and let $g : A \to B$ be VWLUD at $x_0 \in A$ and $f : B \to \mathcal{N}$ be VWLUD at $g(x_0) \in B$. Then $f \circ g : A \to \mathcal{N}$ is VWLUD at x_0.*

Proof. Since g is VWLUD at x_0, there exists an open neighborhood Ω_g of x_0 in A such that g is continuous on Ω_g. Similarly, there exists an open neighbourhood Ω_f of $g(x_0)$ in B such that f is continuous on Ω_f. Let

$$\Omega = \Omega_g \cap g^{-1}(\Omega_f).$$

Then Ω is an open neighborhood of x_0 in A and $g(\Omega) \subset \Omega_f$. Thus g is continuous on Ω and f is continuous on $g(\Omega)$, and hence $f \circ g$ is continuous on Ω. Now, let $\epsilon > 0$ in $\mathcal{N}$ be given. As g is VWLUD at x_0, there exists $\delta_g > 0$ in $\mathcal{N}$ such that $B(x_0, \delta_g) \subset \Omega$ and

$$|g(y) - g(x) - g'(x_0)(y-x)| \le \frac{\epsilon}{2(|f'(g(x_0))| + 1)}|y-x|$$

for all $x, y \in B(x_0, \delta_g)$. Likewise, as f is VWLUD at $g(x_0)$, there exists $\delta_f > 0$ in $\mathcal{N}$ such that $B(g(x_0), \delta_f) \subset \Omega_f$ and

$$|f(y) - f(x) - f'(g(x_0))(y-x)| \le \frac{\epsilon}{2(|g'(x_0)| + 1)}|y-x|$$

and for all $x, y \in B(g(x_0), \delta_f)$.

As g is continuous at x_0, there exists $\delta_c > 0$ in $\mathcal{N}$ such that $B(x_0, \delta_c) \subset \Omega$ and $|g(x) - g(x_0)| < \delta_f$ for all $x \in B(x_0, \delta_c)$.

Finally, as g is VWLUD at x_0, there exists $\delta_0 > 0$ in $\mathcal{N}$ such that $B(x_0, \delta_0) \subset \Omega$,

$$|g(y) - g(x) - g'(x_0)(y-x)| \le |y-x|$$

for all $x, y \in B(x_0, \delta_0)$, and hence

$$|g(y) - g(x)| \le |y-x| + |g'(x_0)||y-x| = (|g'(x_0)| + 1)|y-x|$$

for all $x, y \in B(x_0, \delta_0)$.

Let $\delta = \min\{\delta_g, \delta_c, \delta_0\}$. Then, $B(x_0, \delta) \subset \Omega$ and, for all $x, y \in B(x_0, \delta)$, we have that

$$\begin{aligned}
&\left|f\big(g(y)\big) - f\big(g(x)\big) - g'(x_0)f'\big(g(x_0)\big)(y-x)\right| \\
&\le \left|f\big(g(y)\big) - f\big(g(x)\big) - f'\big(g(x_0)\big)\big(g(y) - g(x)\big)\right| + \left|f'\big(g(x_0)\big)\right| \left|g(y) - g(x) - g'(x_0)(y-x)\right| \\
&\le \frac{\epsilon}{2(|g'(x_0)| + 1)}|g(y) - g(x)| + \frac{\left|f'\big(g(x_0)\big)\right|}{2(\left|f'\big(g(x_0)\big)\right| + 1)}\epsilon|y-x| \\
&\le \epsilon|y-x|.
\end{aligned}$$

□

4. VWLUD^n FUNCTIONS AND THEIR PROPERTIES

Just as WLUD^n extended the WLUD concept to higher orders of differentiability in [1], we now define VWLUD^n functions with the goal to later obtain a weaker sufficient criterion for the mean value theorem for functions over $\mathcal{N}$ than that in [1].

Definition 4.1 (WLUD^n). *Let $A \subseteq \mathcal{N}$ be open and let $f : A \to \mathcal{N}$. Given $n \in \#N$, we say that f is WLUD^n at $x_0 \in A$ if there is an open neighbourhood Ω of x_0 in A in which f is n times differentiable, and if for every $\epsilon > 0$ in $\mathcal{N}$ there exists $\delta > 0$ in $\mathcal{N}$ such that $B(x_0, \delta) \subset \Omega$ and*

$$x, y \in B(x_0, \delta) \implies \left|f(y) - \sum_{k=0}^{n} \frac{f^{(k)}(x)}{k!}(y-x)^k\right| \le \epsilon|y-x|^n.$$

Definition 4.2 (VWLUD^n). *Let $A \subseteq \mathcal{N}$ be open and let $f : A \to \mathcal{N}$. Given $n \in \#N$, we say that f is VWLUD^n at $x_0 \in A$ if there is an open neighbourhood Ω of x_0 in A in which f is C^{n-1}, if $f^{(n)}(x_0)$ exists, and if for every $\epsilon > 0$ in $\mathcal{N}$ there exists $\delta > 0$ in $\mathcal{N}$ such that $B(x_0, \delta) \subset \Omega$ and*

$$x, y \in B(x_0, \delta) \implies \left| f(y) - \sum_{k=0}^{n-1} \frac{f^{(k)}(x)}{k!}(y-x)^k - \frac{f^{(n)}(x_0)}{n!}(y-x)^n \right| \leq \epsilon |y-x|^n.$$

As WLUD^2 functions are C^2 [1], it is easily seen that a function which is WLUD^2 at x_0 is also VWLUD^2 at x_0. We have seen that the fact that a VWLUD function might only be differentiable at a single point is what differentiates this class of functions from the class of WLUD functions. The exact definition of the VWLUD^n criterion arises from a very similar idea, though we will focus on the $n = 2$ case, as this is all that is needed for the mean value theorem. To show that WLUD^2 functions are VWLUD^2, we first define a weaker second order continuity characterization analogous to the WC^1 condition in the first order.

Definition 4.3 (WC^2). *Let $A \subseteq \mathcal{N}$ be open and let $f : A \to \mathcal{N}$. Given $x_0 \in A$, we say that f is weakly C^2 (WC^2) at x_0 if f is twice-differentiable at x_0 and if for every $\epsilon > 0$ in $\mathcal{N}$ there exists $\delta > 0$ in $\mathcal{N}$ such that*

$$(x \in B(x_0, \delta) \wedge f''(x) \text{ exists}) \implies |f''(x) - f''(x_0)| < \epsilon.$$

Proposition 4.4. *Let $A \subseteq \mathcal{N}$ be open and let $f : A \to \mathcal{N}$ be VWLUD^2 at $x_0 \in A$. Then f is WC^2 at x_0.*

Proof. Let $\epsilon > 0$ in $\mathcal{N}$ be given. Then there exists $\delta > 0$ in $\mathcal{N}$ such that $B(x_0, \delta) \subset A$, f is C^1 on $B(x_0, \delta)$ and if $x, y \in B(x_0, \delta)$ then

$$\left| f(y) - f(x) - f'(x)(y-x) - \frac{1}{2} f''(x_0)(y-x)^2 \right| \leq \frac{\epsilon}{3}(y-x)^2.$$

Let $x \in B(x_0, \delta)$ be given with f twice-differentiable at x. As $f''(x)$ exists, we can find a $\delta_1 > 0$ in $\mathcal{N}$, $\delta_1 < \delta$, such that if $y \in B(x, \delta_1)$ and $y \neq x$ then

$$\left| \frac{f'(y) - f'(x)}{y - x} - f''(x) \right| < \frac{\epsilon}{3}.$$

Choose a $y \in B(x, \delta_1)$ with $y \neq x$. Then,

$$\begin{aligned} \left| f''(x) - f''(x_0) \right| &\leq \left| \frac{f(y) - f(x)}{(y-x)^2} - \frac{f'(x)}{y-x} - \frac{1}{2} f''(x_0) \right| + \left| f''(x) - \frac{f'(y) - f'(x)}{y-x} \right| \\ &\quad + \left| \frac{f(x) - f(y)}{(x-y)^2} - \frac{f'(y)}{x-y} - \frac{1}{2} f''(x_0) \right| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

□

Remark 4.5. *From the above proposition, we observe that if f is VWLUD^2 at x_0 and twice-differentiable in a neighbourhood of x_0, then f is C^2 at x_0. It follows that f is then WLUD^2 at x_0. For this reason, the definition of VWLUD^n at a point x_0 allows for functions which are not necessarily n times differentiable in a neighbourhood of x_0; otherwise, the VWLUD^n and WLUD^n criteria would coincide for the $n = 2$ case.*

Proposition 4.6. *Let $A \subseteq \mathcal{N}$ be open and let $f : A \to \mathcal{N}$ be VWLUD^n at $x_0 \in A$ with $n \geq 2$. Then f is VWLUD^{n-1} at x_0.*

Proof. First note that, since f is VWLUD^n at x_0, f is n times differentiable at x_0 and there exists an open neighbourhood Ω of x_0 in A such that f is C^{n-1} on Ω. In particular, f is $(n-1)$ times differentiable at x_0 and C^{n-2} on Ω. Now let $\epsilon > 0$ in $\mathcal{N}$ be given. Then there exists $\delta_1 > 0$ in $\mathcal{N}$ such that $B(x_0, \delta_1) \subset \Omega$ and

$$\left| f(y) - \sum_{k=0}^{n-1} \frac{f^{(k)}(x)}{k!}(y-x)^k - \frac{f^{(n)}(x_0)}{n!}(y-x)^n \right| \leq \frac{\epsilon}{3}|y-x|^n$$

for all $x, y \in B(x_0, \delta_1)$. Moreover, since f is C^{n-1} at x_0, there exists $\delta_2 > 0$ in $\mathcal{N}$ such that $B(x_0, \delta_2) \subset \Omega$ and

$$\left| f^{(n-1)}(x) - f^{(n-1)}(x_0) \right| < \frac{\epsilon}{3}(n-1)!$$

for all $x \in B(x_0, \delta_2)$.

Let $\delta = \min\left\{\delta_1, \delta_2, \frac{1}{2}, \frac{n!\epsilon}{6(|f^{(n)}(x_0)|+1)}\right\}$. Then $B(x_0, \delta) \subset \Omega$ and, for all $x, y \in B(x_0, \delta)$, we have that

$$\begin{aligned}
&\left| f(y) - \sum_{k=0}^{n-2} \frac{f^{(k)}(x)}{k!}(y-x)^k - \frac{f^{(n-1)}(x_0)}{(n-1)!}(y-x)^{n-1} \right| \\
&\leq \left| f(y) - \sum_{k=0}^{n-1} \frac{f^{(k)}(x)}{k!}(y-x)^k - \frac{f^{(n)}(x_0)}{n!}(y-x)^n \right| \\
&\quad + \left| \frac{f^{(n-1)}(x)}{(n-1)!}(y-x)^{n-1} - \frac{f^{(n-1)}(x_0)}{(n-1)!}(y-x)^{n-1} \right| + \left| \frac{f^{(n)}(x_0)}{n!}(y-x)^n \right| \\
&\leq \frac{\epsilon}{3}|y-x||y-x|^{n-1} + \frac{\epsilon}{3}|y-x|^{n-1} + \frac{|f^{(n)}(x_0)|}{n!}|y-x||y-x|^{n-1} \\
&\leq \frac{\epsilon}{3}|y-x|^{n-1} + \frac{\epsilon}{3}|y-x|^{n-1} + \frac{\epsilon}{3}|y-x|^{n-1} \\
&= \epsilon|y-x|^{n-1}.
\end{aligned}$$

□

Proposition 4.7. *Let $A \subseteq \mathcal{N}$ be open and let $f : A \to \mathcal{N}$ be VWLUD^2 at $x_0 \in A$. Then f' is VWLUD at x_0.*

Proof. Since f is VWLUD^2 at x_0, f is twice differentiable at x_0 and there is an open neighbourhood Ω of x_0 in A such that f is C^1 on Ω. Thus, f' is differentiable at x_0 and is continuous on Ω. Now, let $\epsilon > 0$ in $\mathcal{N}$ be given. Then, as f is VWLUD^2 at x_0, there exists $\delta > 0$ in $\mathcal{N}$ such that $B(x_0, \delta) \subset \Omega$ and

$$\left| f(y) - f(x) - f'(x)(y-x) - \frac{1}{2}f''(x_0)(y-x)^2 \right| \leq \frac{\epsilon}{2}(y-x)^2$$

for all $x, y \in B(x_0, \delta)$. It follows that

$$\left| \frac{f(y)-f(x)}{y-x} - f'(x) - \frac{1}{2}f''(x_0)(y-x) \right| \leq \frac{\epsilon}{2}|y-x|$$

for all distinct $x, y \in B(x_0, \delta)$.

Thus, for all $x \neq y$ in $B(x_0, \delta)$, we have that

$$\begin{aligned}
&|f'(y) - f'(x) - f''(x_0)(y-x)| \\
&\leq \left| \frac{f(y)-f(x)}{y-x} - f'(x) - \frac{1}{2}f''(x_0)(y-x) \right| + \left| \frac{1}{2}f''(x_0)(x-y) + f'(y) - \frac{f(y)-f(x)}{y-x} \right| \\
&= \left| \frac{f(y)-f(x)}{y-x} - f'(x) - \frac{1}{2}f''(x_0)(y-x) \right| + \left| \frac{f(x)-f(y)}{x-y} - f'(y) - \frac{1}{2}f''(x_0)(x-y) \right|
\end{aligned}$$

$$\leq \frac{\epsilon}{2}|y-x| + \frac{\epsilon}{2}|y-x| = \epsilon|y-x|.$$

Altogether, f' is differentiable at x_0 and is continuous on the open neighbourhood Ω of x_0 in A; moreover, for all $\epsilon > 0$ in $\mathcal{N}$ there exists $\delta > 0$ in $\mathcal{N}$ such that $B(x_0, \delta) \subset \Omega$ and

$$|f'(y) - f'(x) - f''(x_0)(y-x)| \leq \epsilon|y-x|$$

for all $x, y \in B(x_0, \delta)$. Thus, f' is VWLUD at x_0. □

5. CALCULUS THEOREMS FOR VWLUD FUNCTIONS

As it was used in the theory of WLUD functions in [1], the following lemma, proved in [14], will be vital in the proof of the inverse function theorem.

Lemma 5.1. *Let $\delta_0 > 0$ in $\mathcal{N}$ be given. Choosing $c \in \mathcal{N}$ with $0 < c \ll 1$, let $\phi : B(0, \delta_0) \to \mathcal{N}$ be such that $|\phi(t)| \leq c|t|$ for all $t \in B(0, \delta_0)$. For all $m \in \#N$, let $\phi^{[m]} = \phi \circ \cdots \circ \phi$ (m times) and set $\phi^{[0]}$ to be the identity map. Let $\delta \in \mathcal{N}$ be such that $0 < \delta \leq (1-c)\delta_0$ and define $\psi : B(0, \delta) \to \mathcal{N}$ by $\psi(t) = \sum_{m=0}^{\infty} \phi^{[m]}(t)$. Then,*

(i) $|\psi(t)| \leq \frac{|t|}{1-c}$*; and*

(ii) $\psi(t) - \phi(\psi(t)) = t$.

Proof. See the proof of Lemma 4.1 in [14]. □

Theorem 5.2 (Inverse Function Theorem)**.** *Let $A \subseteq \mathcal{N}$ be open and let $f : A \to \mathcal{N}$ be VWLUD at some $x_0 \in A$ with $f'(x_0) \neq 0$. Let $y_0 := f(x_0)$. Then there exist $\delta, \eta > 0$ in $\mathcal{N}$ and a function $F : B(y_0, \eta) \to \mathcal{N}$ such that*

(i) $B(x_0, \delta) \subseteq A$*;*

(ii) $f|_{B(x_0,\delta)}$ *is injective;*

(iii) $B(y_0, \eta) \subseteq f(B(x_0, \delta))$ *and* $F(B(y_0, \eta)) \subseteq B(x_0, \delta)$*;*

(iv) $f(F(x)) = x$ *for all* $x \in B(y_0, \eta)$*; and*

(v) F *is VWLUD at* y_0 *and* $F'(y_0) = 1/(f' \circ F)(y_0) = 1/f'(x_0)$.

Proof. Without loss of generality, we may assume that $x_0 = 0$ and $y_0 = 0$, otherwise we can instead consider the function $\tilde{f}(x) = f(x + x_0) - y_0$. If $f'(x_0) < 0$ we can instead apply this proof to $(-f)$, so we may additionally assume that $f'(x_0) > 0$.

Since f is VWLUD at $x_0 = 0$, there is a neighbourhood Ω of 0 in A such that f is continuous on Ω. Let $L = f'(0)$ and let $\phi(x) = x - \frac{1}{L}f(x)$. Then ϕ is continuous on Ω and VWLUD at 0 with $\phi'(0) = 1 - \frac{1}{L}f'(0) = 0$. Let $c \in \mathcal{N}$ be such that $0 < c \ll 1$. Then there exists $\delta_1 > 0$ in $\mathcal{N}$ such that $B(0, \delta_1) \subseteq \Omega$ and, for all $s, t \in B(0, \delta_1)$, we have that

$$|\phi(s) - \phi(t) - \phi'(0)(s-t)| \leq c|s-t|,$$

and hence

$$|\phi(s) - \phi(t)| \leq c|s-t|.$$

Setting $t = 0$, we obtain that $|\phi(s)| \leq c|s|$ for all $s \in B(0, \delta_1)$. Moreover, if $f(s) = f(t)$, then

$$|s-t| = \left|\left(s - \frac{1}{L}f(s)\right) - \left(t - \frac{1}{L}f(t)\right)\right| = |\phi(s) - \phi(t)| \leq c|s-t|,$$

so we must have that $s = t$ since $c \ll 1$. That is, f is injective on $B(0, \delta_1)$.

Since f is VWLUD at 0, there exists $\delta_2 > 0$ in $\mathcal{N}$ such that $B(0,\delta_2) \subseteq \Omega$ and, for all $s,t \in B(0,\delta_2)$ we have that

$$|f(s) - f(t) - L(s-t)| \leq \frac{L}{2}|s-t|.$$

Let $\delta = \min\{(1-c)\delta_1, \delta_2\}$. Then $f|_{B(0,\delta)}$ is injective as $B(0,\delta) \subset B(0,\delta_1) \subset A$. So, conditions (i) and (ii) are satisfied.

As $\delta \leq (1-c)\delta_1$, there is a function ψ defined on $B(0,\delta)$ with the properties described in Lemma 1. Let $\eta = L(1-c)\delta$ and define $F(x) = \psi\left(\frac{x}{L}\right)$ for every $x \in B(0,\eta)$. Then, for every $x \in B(0,\eta)$, since

$$\left|\frac{x}{L}\right| < \frac{L(1-c)\delta}{L} = (1-c)\delta < \delta,$$

we have $\frac{x}{L} \in B(0,\delta)$. So, property (i) of Lemma 1 allows us to establish that

$$|F(x)| = \left|\psi\left(\frac{x}{L}\right)\right| \leq \frac{|x|}{L(1-c)} < \frac{\eta}{L(1-c)} = \delta.$$

Thus, $F(B(0,\eta)) \subseteq B(0,\delta)$. We note that for every $x \in B(0,\delta)$ we have that

$$x - \phi(x) = x - \left(x - \frac{1}{L}f(x)\right) = \frac{f(x)}{L},$$

hence

$$\phi(x) = x - \frac{f(x)}{L}. \tag{5.1}$$

Moreover, as $\frac{x}{L} \in B(0,\delta)$ for any $x \in B(0,\eta)$, property (ii) of Lemma 1 gives us that

$$\psi\left(\frac{x}{L}\right) - \phi\left(\psi\left(\frac{x}{L}\right)\right) = \frac{x}{L}.$$

Substituting Equation (5.1) into the above equation with $\psi\left(\frac{x}{L}\right)$ in place of x, we observe that

$$\psi\left(\frac{x}{L}\right) - \left[\psi\left(\frac{x}{L}\right) - \frac{1}{L}f\left(\psi\left(\frac{x}{L}\right)\right)\right] = \frac{x}{L},$$

thus

$$f(F(x)) = f\left(\psi\left(\frac{x}{L}\right)\right) = x$$

for all $x \in B(0,\eta)$. This shows (iv).

Given $y \in B(0,\eta)$, let $x = F(y)$. Then $x \in B(0,\delta)$ and $f(x) = f(F(y)) = y$, so $y \in f(B(0,\delta))$. Thus, $B(0,\eta) \subseteq f(B(0,\delta))$, which completes the proof of (iii).

Now, for any $s,t \in B(0,\delta)$, we have

$$|f(s) - f(t) - L(s-t)| \leq \frac{L}{2}|s-t|,$$

and it follows that

$$|f(s) - f(t)| \geq L|s-t| - \frac{L}{2}|s-t| = \frac{L}{2}|s-t|. \tag{5.2}$$

Let $x, y \in B(0,\eta)$ be given, and let $t_x = F(x)$ and $t_y = F(y)$. Since $t_x, t_y \in B(0,\delta)$, we can apply Equation (5.2) to obtain

$$|F(y) - F(x)| = |t_y - t_x| \leq \frac{2}{L}|f(t_y) - f(t_x)| = \frac{2}{L}|y - x|.$$

Thus, F is (Lipschtiz) continuous on the open neighbourhood $B(0,\eta)$ of $y_0 = 0$.

Now let $\epsilon > 0$ in $\mathcal{N}$ be given. Since f is VWLUD at $x_0 = 0$ with $f'(0) = L$, there exists $\delta_3 \in \mathcal{N}$ such that $0 < \delta_3 < \delta$ and, for all $t_1, t_2 \in B(0, \delta_3)$, we have that

$$|f(t_2) - f(t_1) - L(t_2 - t_1)| \le \frac{L^2}{2}\epsilon|t_2 - t_1|.$$

Let $V = F^{-1}(B(0, \delta_3))$. Then V is an open neighbourhood of $y_0 = 0$ in $B(0, \eta)$, by continuity of F on $B(0, \eta)$. Thus, there exists $\eta_\epsilon > 0$ in $\mathcal{N}$ such $B(0, \eta_\epsilon) \subset V \subset B(0, \eta)$. It follows that, for all $x, y \in B(0, \eta_\epsilon)$, we have $t_x := F(x) \in B(0, \delta_3)$ and $t_y := F(y) \in B(0, \delta_3)$. Thus, for all $x, y \in B(0, \eta_\epsilon)$, we have that

$$\begin{aligned}\left|F(y) - F(x) - \frac{1}{f'(0)}(y - x)\right| &= \frac{1}{f'(0)}|y - x - f'(0)(F(y) - F(x))| \\ &= \frac{1}{L}|f(t_y) - f(t_x) - L(t_y - t_x)| \\ &\le \frac{1}{L}\frac{L^2}{2}\epsilon|t_y - t_x| \le \frac{1}{L}\frac{L^2}{2}\epsilon\frac{2}{L}|y - x| \\ &= \epsilon|y - x|.\end{aligned}$$

This shows that F is VWLUD at $y_0 = 0$ with

$$F'(y_0) = \frac{1}{f'(0)} = \frac{1}{f'(x_0)} = \frac{1}{(f' \circ F)(y_0)},$$

and finishes the proof of (v). □

Theorem 5.3 (Local Intermediate Value Theorem). *Let $A \subseteq \mathcal{N}$ be open and let $f : A \to \mathcal{N}$ be VWLUD at $x_0 \in A$ with $f'(x_0) \neq 0$. Then there exists a neighbourhood U of x_0 such that f has the intermediate value property on U. That is, given $a, b \in U$ with $a < b$, if c is between $f(a)$ and $f(b)$, then there exists $x \in (a, b)$ such that $f(x) = c$.*

Proof. Without loss of generality, we may assume that $f'(x_0) > 0$, else we can instead consider $-f$ in place of f. Let $\delta > 0$ and $\eta > 0$ be as in the proof of the inverse function theorem. Then f is continuous and injective on $B(x_0, \delta) \subset A$ and, for all $x, y \in B(x_0, \delta)$, we have that

$$|f(y) - f(x) - f'(x_0)(y - x)| \le \frac{f'(x_0)}{2}|y - x|.$$

It follows that

$$\frac{f(y) - f(x)}{y - x} \ge f'(x_0) - \frac{f'(x_0)}{2} = \frac{f'(x_0)}{2} > 0,$$

for all distinct $x, y \in B(x_0, \delta)$ and hence f is strictly increasing on $B(x_0, \delta)$.

Using the proof of the inverse function theorem, we have that $B(f(x_0), \eta) \subseteq f(B(x_0, \delta))$. Let $U = f^{-1}(B(f(x_0), \eta))$. Then $U \subseteq B(x_0, \delta)$ and U is open since f is continuous on $B(x_0, \delta)$.

Let $a, b \in U$ with $a < b$ and let $c \in (f(a), f(b))$ be given. Then $c \in B(f(x_0), \eta)$ and, by definition of U, there exists $x \in U$ such that $f(x) = c$. Moreover, $x \in (a, b)$ as f is strictly increasing on U. □

Theorem 5.4 (Local Mean Value Theorem). *Let $A \subseteq \mathcal{N}$ be open and let $f : A \to \mathcal{N}$ be VWLUD^2 at some $x_0 \in A$ with $f''(x_0) \neq 0$. Then there exists a neighbourhood U of x_0 in A such that f has the mean value property on U. That is, given $a, b \in U$ with $a < b$, there exists $c \in (a, b)$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Proof. As f is VWLUD^2 at x_0, there exists $\delta_1 > 0$ in $\mathcal{N}$ such that $B(x_0, \delta_1) \subseteq A$ and

$$\left| f(y) - f(x) - f'(x)(y-x) - \frac{1}{2} f''(x_0)(y-x)^2 \right| \leq \frac{1}{4} f''(x_0)(y-x)^2$$

for all $x, y \in B(x_0, \delta_1)$. As in the proof of the local intermediate value theorem, we may assume, without loss of generality, that $f''(x_0) > 0$. Thus, for all $x \neq y$ in $B(x_0, \delta_1)$, we have that

$$f(y) - f(x) - f'(x)(y-x) \geq \left(\frac{1}{2} f''(x_0) - \frac{1}{4} f''(x_0) \right) (y-x)^2 = \frac{1}{4} f''(x_0)(y-x)^2 > 0. \tag{5.3}$$

Moreover, f' is VWLUD at x_0 since f is VWLUD^2 at x_0, so by the local intermediate value theorem, there exists $\delta_2 > 0$ in $\mathcal{N}$ such that $B(x_0, \delta_2) \subseteq A$ and f' has the intermediate value property on $B(x_0, \delta_2)$. Let $\delta = \min\{\delta_1, \delta_2\}$, let $U = B(x_0, \delta)$, and let $a, b \in U$ with $a < b$. It follows from Equation (5.3), with $y = b$ and $x = a$, that $f(b) > f(a) + f'(a)(b-a)$; and hence

$$f'(a) < \frac{f(b) - f(a)}{b - a}.$$

Using the same equation, with $y = a$ and $x = b$, we obtain that $f(a) > f(b) + f'(b)(a-b)$; and hence

$$f'(b) > \frac{f(b) - f(a)}{b - a}.$$

Thus, by the intermediate value theorem applied to f', there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

□

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