On Non-Archimedean Valued Fields: A Survey of Algebraic, Topological and Metric Structures, Analysis and Applications

Khodr Shamseddine and Angel Barría Comicheo

Abstract In this survey paper, we first briefly review basic properties of ultrametric spaces, valued fields and ordered fields as well as the connection between these different mathematical objects. As examples, we introduce the so-called general Hahn fields and Levi-Civita fields, and we present a summary of their key properties. Then, for the rest of the paper, we focus our attention on two special Levi-Civita fields: $\mathcal{R}$ and its complex counterpart $\mathcal{C}$. Among all the non-Archimedean fields surveyed in the first part of the paper, $\mathcal{R}$ and $\mathcal{C}$ are unique from a pure Mathematics point of view: $\mathcal{R}$ (respectively, $\mathcal{C}$) is the smallest non-Archimedean valued field extension of the field of real numbers $\mathbb{R}$ (respectively, the field of complex numbers $\mathbb{C}$) that is real closed (respectively, algebraically closed) and Cauchy-complete in the valuation topology. Moreover, because of the left-finiteness of the supports of the Levi-Civita numbers, those numbers can be used on a computer, thus allowing for many useful computational applications. We review some of our research work on $\mathcal{R}$ and $\mathcal{C}$ as well as on the spaces $\mathcal{R}^2$ and $\mathcal{R}^3$: one-dimensional and multi-dimensional calculus, power series and analytic functions, measure theory and integration, unconstrained and constrained optimization, operator theory on the Banach space c_0 of null sequences of elements of $\mathcal{C}$, and computational applications.

Keywords Non-Archimedean valued fields · Ultrametric spaces · Hahn fields · Levi-Civita fields · Non-Archimedean analysis · Non-Archimedean calculus · Computational applications

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K. Shamseddine (✉)
Department of Physics and Astronomy, University of Manitoba, Winnipeg, MB, Canada
e-mail: Khodr.Shamseddine@umanitoba.ca

A. Barría Comicheo
Department of Mathematics, University of Manitoba, Winnipeg, MB, Canada

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1 Introduction

In the first part of the paper, we provide all the necessary preliminaries needed to introduce the Levi-Civita fields in a general context considering their algebraic, metric and ordered structures. We begin by reviewing the properties of ultrametric spaces and non-Archimedean valued fields in Sect. 2, establishing a close correspondence between the two mathematical concepts and introducing the non-Archimedean notion of spherical completeness and related results. Then, in Sect. 3, we introduce the Hahn fields and review their role among ordered fields with the same "level of non-Archimedicity", which is a generalization of the role that the field of real numbers $\mathbb{R}$ plays among all Archimedean ordered fields. Finally, we discuss the Levi-Civita fields as particular subfields of the Hahn fields.

In the second part of the paper, we focus our attention on two particular Levi-Civita fields: $\mathcal{R}$ and $\mathcal{C} := \mathcal{R} \oplus i\mathcal{R}$. $\mathcal{R}$ (resp. $\mathcal{C}$) is the smallest non-Archimedean valued field extension of $\mathbb{R}$ (resp. $\mathbb{C}$) that is real closed (resp. algebraically closed) and Cauchy-complete in the valuation topology. After reviewing the algebraic and topological structures of $\mathcal{R}$ and $\mathcal{C}$ in Sect. 4, we review our work on developing calculus on $\mathcal{R}$ and $\mathcal{R}^n$, showing in Sect. 5 that, for the so-called weakly locally uniformly differentiable (WLUD) functions at a point or on an open subset of $\mathcal{R}$ or $\mathcal{R}^n$, the important theorems of real calculus hold locally. Then we summarize in Sect. 6 the convergence and analytical properties of power series, showing that they have the same smoothness behavior as real and complex power series. Moreover, we present in Sect. 7 a Lebesgue-like measure and integration theory on $\mathcal{R}$, $\mathcal{R}^2$ and $\mathcal{R}^3$ with applications of that theory. As well, we discuss in Sect. 8 solutions to one-dimensional and multi-dimensional optimization problems based on continuity and differentiability concepts that are stronger than the topological ones. Then, in Sect. 9, we review some of the computational applications of the Levi-Civita numbers which can be used on a computer because of the left-finiteness of the supports of those numbers. Finally, in Sect. 10, we present a quick review of our work on developing an operator theory on the Banach space c_0 of null sequences of $\mathcal{C}$.

2 Preliminaries

2.1 Non-Archimedean Valued Fields

We start this subsection with the definition of a valuation on a field.

Definition 2.1 Let K be a field. A *valuation* on K is a map $|\ \ | : K \to \mathbb{R}$ satisfying the following properties, for all $x, y \in K$:

(1) $|x| \geq 0$, and $|x| = 0$ if and only if $x = 0$,
(2) $|xy| = |x||y|$,
(3) $|x + y| \leq |x| + |y|$.

The pair $(K, |\ \ |)$ is called a valued field.

It is not hard to see that $|1_K| = 1$, $|-x| = |x|$ and $|x^{-1}| = |x|^{-1}$ for $x \neq 0$. In the rest of the article we will denote the set $K \setminus \{0\}$ by K^*.

A valuation $|\ \ |$ on K is called *non-Archimedean* if it satisfies the *strong triangle inequality*: $|x + y| \leq \max\{|x|, |y|\}$ for all $x, y \in K$. Otherwise it is called *Archimedean*.

Theorem 2.2 ([55, 1.1], [33, lemma 8.2]) *Let $(K, |\ \ |)$ be a valued field. The following conditions are equivalent.*

(1) *$|\ \ |$ is non-Archimedean.*
(2) *If $a, b \in K$ and $|a| < |b|$, then $|b - a| = |b|$ (isosceles triangle principle).*
(3) *The set $\{|n1_K| : n \in \mathbb{N}\}$ is bounded.*
(4) *$|n1_K| \leq 1$ for every $n \in \mathbb{N}$.*
(5) $|2 \cdot 1_K| \leq 1$.

As the reader can deduce from the definition, the features of a valued field will depend on the algebraic properties of the field (whether it is algebraically closed, or real-closed, etc.) and the properties of the valuation (whether it is Archimedean or non-Archimedean, or whether $|K^*|$ is discrete or dense in $(0, \infty)$, etc.). As in the real and complex cases, the valuation defines a metric $\Delta(x, y) := |x - y|$ for $x, y \in K$, which allows us to consider convergence of sequences and series, and continuity of functions on K. Sometimes, the valued field will admit an order compatible with the field operations which enriches the properties of the valued field, and generates an order topology. It is interesting to ask whether the order topology coincides with the topology induced by the valuation. In [4] and [12] it is shown that for every ordered field, there exists a valuation that induces the order topology of the field, although sometimes the codomain of the valuation needs to be an ordered group that cannot be embedded in the real numbers. Those latter valuations are called Krull valuations or general valuations and are studied in [12].

2.2 Ultrametric Spaces

The importance of non-Archimedean valued fields relies on the alternative models they provide compared to the commonly studied models defined using real or complex numbers. How different the non-Archimedean structures can be from the real or complex ones depends in a major part on how different the underlying non-Archimedean valued field is from the real or complex number fields.

The first aspect of non-Archimedean valued fields that we will review is their metric structure, e.g. how different the convergence criteria for sequences and series can be from the ones in the real or complex numbers fields. For this, we will leave the algebraic structure of the valued field aside, and we will focus on its properties as a metric space leading to the notion of an ultrametric space.

Recall that a *metric* on a set X is a function $\Delta : X \times X \to \mathbb{R}$ satisfying the following properties for all $x, y, z \in X$:

(1) $\Delta(x, y) \geq 0$, and $\Delta(x, y) = 0$ if and only if $x = y$,
(2) $\Delta(x, y) = \Delta(y, x)$,
(3) $\Delta(x, y) \leq \Delta(x, z) + \Delta(z, y)$ (triangle inequality).

The pair (X, Δ) is called a *metric space*. In particular, when the metric satisfies the so-called *strong triangle inequality* $\Delta(x, y) \leq \max\{\Delta(x, z), \Delta(z, y)\}$ for all $x, y, z \in X$, the pair (X, Δ) is called an *ultrametric space*.

Any subset of a non-Archimedean valued field $(K, |\ |)$ with the map $(x, y) \mapsto |x - y|$ constitutes an ultrametric space. Notice that with this example we have listed all ultrametric spaces, since W. Schikhof proved in [31] that any ultrametric space can be isometrically embedded into a non-Archimedean valued field.

When a metric satisfies the strong triangle inequality, geometrical situations occur that do not occur otherwise. As an example of such unusual situations, we will consider triangles in an ultrametric space. Let (X, Δ) be a metric space. The following condition is called the *isosceles triangle principle*: for all $x, y, z \in X$, if $\Delta(x, z) \neq \Delta(z, y)$ then $\Delta(x, y) = \max\{\Delta(x, z), \Delta(z, y)\}$; that is, every triangle with vertices in X is isosceles.

Theorem 2.3 ([28, p. 3], [55, 2.A]) *Let (X, Δ) be a metric space. The metric Δ is an ultrametric if and only if it satisfies the isosceles triangle principle.*

Before we continue, let us present some important notations.

Notation 2.4 Let (X, Δ) be a metric space and let $a \in X$ and $r > 0$. The sets $B(a, r) := \{x \in X : \Delta(x, a) < r\}$ and $B[a, r] := \{x \in X : \Delta(x, a) \leq r\}$ are called the *open and closed balls of center a and radius r*, respectively. The family of open balls forms a base of neighbourhoods for a uniquely determined Hausdorff topology on X. This topology is called the *topology induced by Δ on X*. With respect to this topology the open balls are open sets and the closed balls are closed sets in X. The *diameter* of a non-empty set $Y \subset X$ is $diam(Y) := \sup\{\Delta(x, y) : x, y \in Y\}$ and the *distance* between two non-empty sets $Y, Z \subset X$ is $dist(Y, Z) := \inf\{\Delta(y, z) : y \in Y, z \in Z\}$. The set of values of a metric $\Delta : X \times X \to \mathbb{R}$ is denoted and defined by $\Delta(X \times X) := \{\Delta(x, y) : x, y \in X\}$.

The following theorem collects the most remarkable results about an ultrametric space, all of which are direct consequences of the strong triangle inequality.

Theorem 2.5 *Let (X, Δ) be an ultrametric space. Then the following properties are satisfied.*

(1) *Each point of a ball is a center of the ball.*
(2) *Each ball in X is both closed and open ("clopen") in the topology induced by the ultrametric.*

(3) *Each ball has an empty boundary.*
(4) *Two balls are either disjoint, or one is contained in the other.*
(5) *Let* $a \in Y \subset X$. *Then* $diam(Y) = \sup\{\Delta(x, a) : x \in Y\}$.
(6) *The radii of a ball* B *form the set* $\{r \in \mathbb{R} : r_1 \leq r \leq r_2\}$, *where* $r_1 = diam(B)$, $r_2 = dist(B, X \backslash B)$ ($r_2 = \infty$ *if* $B = X$). *It may happen that* $r_1 < r_2$, *so that a ball may have infinitely many radii.*
(7) *If two balls* B_1, B_2 *are disjoint, then* $dist(B_1, B_2) = \Delta(x, y)$ *for all* $x \in B_1, y \in B_2$.
(8) *Let* $U \neq \emptyset$ *be an open subset of* X. *Given a sequence* $(r_n)_n$ *in* $(0, \infty)$, *strictly decreasing and convergent to* 0, *then there exists a partition of* U *formed by balls of the form* $B[a, r_n]$, *with* $a \in U$ *and* $n \in \mathbb{N}$.
(9) *Let* $\varepsilon \in \mathbb{R}^+$. *For* $x, y \in X$, *the relation* $\Delta(x, y) < \varepsilon$ *is an equivalence relation and induces a partition of* X *into open balls of radius* ε. *Analogously for* $\Delta(x, y) \leq \varepsilon$ *and closed balls.*
(10) *Let* $Y \subset X$, B *a ball in* X, $B \cap Y \neq \emptyset$. *Then,* $B \cap Y$ *is a ball in* Y.
(11) *Let* $(x_n)_n$ *be a sequence in* X *converging to* $x \in X$, *then for each* $a \in X \setminus \{x\}$, *there exists* $N \in \mathbb{N}$ *such that* $\Delta(x_n, a) = \Delta(x, a)$ *for all* $n \geq N$.
(12) *There are no new values of an ultrametric after completion, i.e. if* $(X^\wedge, \Delta^\wedge)$ *is the completion of* (X, Δ), *then* $\Delta(X \times X) = \Delta^\wedge(X^\wedge \times X^\wedge)$.
(13) *A sequence* $(x_n)_n$ *on* X *is Cauchy if and only if* $\lim_{n\to\infty} \Delta(x_n, x_{n+1}) = 0$.

Proof The property (8) can be found in [33, Theorem 18.6], while (5) is in [32, 1.D]. The property (3) follows directly from (2) and the proofs of the remaining can be found in [28, pp. 3–4]. □

For results regarding compactness and separability of ultrametric spaces, and for results about ultrametrizability of a topological space, we refer the reader to [12].

2.3 Spherical Completeness

Recall that a metric space is said to be Cauchy complete if every Cauchy sequence is convergent or, equivalently, if each nested sequence of closed balls whose radii form a null sequence has a non-empty intersection. This motivates the following definition.

Definition 2.6 An ultrametric space is called *spherically complete* if each nested sequence of balls has a non-empty intersection.

Remark 2.7 The concept of spherical completeness plays a key role as a necessary and sufficient condition for the validity of the Hahn-Banach theorem in the non-Archimedean context (see [55, 4.10, 4.15]). Furthermore, spherically complete spaces satisfy important properties as we will see below: a fixed point theorem and best approximations.

It is clear that a spherically complete ultrametric space is Cauchy complete, but the converse is not always true as we will see when we review the Levi-Civita fields.

Nevertheless, the following lemma is a partial converse.

Lemma 2.8 ([35, Lemma 1.7]) *Suppose that* (X, Δ) *is a Cauchy complete ultrametric space. If* 0 *is the only accumulation point of the set* $\Delta(X \times X)$*, then* (X, Δ) *is spherically complete.*

The concept of spherical completeness is geometrical rather than topological.

Theorem 2.9 ([55, 2.F]) *Let* (X, Δ) *be a complete ultrametric space. Then the formula*

$$\sigma(x, y) := \inf\{2^{-n} : n \in \mathbb{Z}, \Delta(x, y) \leq 2^{-n}\}$$

defines an ultrametric σ *such that* $\Delta \leq \sigma \leq 2\Delta$*, and* (X, σ) *is spherically complete.*

One of the attributes of spherically complete ultrametric spaces is that they satisfy a stronger version of the fixed point theorem for complete metric spaces.

Definition 2.10 Let (X, Δ) be a metric space. A function $f : X \to X$ is called a *shrinking map* when $\Delta(f(x), f(y)) < \Delta(x, y)$ for all $x, y \in X$, $x \neq y$. If there exists $k \in (0, 1)$ such that $\Delta(f(x), f(y)) < k\Delta(x, y)$ for all $x, y \in X$, $x \neq y$, then f is called a *contraction*.

The fixed point theorem for complete metric spaces states that every contraction of a complete metric space has a unique fixed point [54, 3.7.4]. However, this theorem cannot be extended to shrinking maps. In fact, the map $f : [1, \infty) \to [1, \infty)$, $f(x) = x + \dfrac{1}{x}$ is a shrinking map defined on a complete space that has no fixed point. In contrast, every shrinking map of a spherically complete ultrametric space has a unique fixed point [29, 2.3].

Definition 2.11 Let Y be a subset of an ultrametric space (X, Δ). Let $a \in X$ and $b \in Y$. Then b is a *best approximation* of a in Y if $\Delta(a, b) = dist(a, Y)$.

Another attribute of a spherically complete ultrametric space is the existence of best approximations as stated in the following result.

Theorem 2.12 ([33, 21.2]) *Let* $Y \neq \emptyset$ *be a spherically complete ultrametric space embedded in an ultrametric space* X*. Then each* $x \in X$ *has a best approximation in* Y*, i.e.* $\min\{\Delta(y, x) : y \in Y\}$ *exists.*

In general, best approximations are not unique.

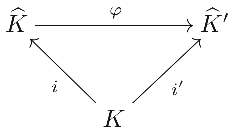
Theorem 2.13 ([33, 21.1]) *Let* $Y \neq \emptyset$ *be a subset of an ultrametric space* X*. Suppose that* Y *has no isolated points. If an element* $a \in X \setminus Y$ *has a best approximation in* Y *then it has infinitely many.*

Note that every non-Archimedean valued field has at least one "spherical completion" and that spherical completion is not always unique [12]. Also, the spherical completeness of a non-Archimdean valued field has been characterized in terms of sequences (pseudo-completeness), and in terms of valued field extensions (maximal completeness). For more information about these equivalent concepts and for a description of the structure of spherically complete valued fields in general, please see [12].

2.4 Completion of Valued Fields

The completion of a valued field as a metric space can be equipped with field operations and a valuation that extend the ones of the original field.

Theorem 2.14 ([14, 1.1.4]) *Let $(K, |\ \ |)$ be a valued field. There exists a Cauchy complete valued field $(\widehat{K}, \widehat{|\ \ |})$, and an embedding $i : K \to \widehat{K}$, such that $|x| = \widehat{|i(x)|}$ for all $x \in K$, and the image $i(K)$ is dense in $\widehat{K}$. If $(\widehat{K}', \widehat{|\ \ |}', i')$ is another such trio, then there exists a unique isomorphism $\varphi : \widehat{K} \to \widehat{K}'$ satisfying $\widehat{|\varphi(x)|}' = \widehat{|x|}$ for all $x \in \widehat{K}$ and making the following diagram commutative:*



Definition 2.15 A pair $(\widehat{K}, \widehat{|\ \ |})$ as in Theorem 2.14 is called a *completion* of the valued field $(K, |\ \ |)$.

Remark 2.16 Let $(K, |\ \ |)$ be a valued field and $(\widehat{K}, \widehat{|\ \ |})$ its completion with embedding $i : K \to \widehat{K}$ such that $|x| = \widehat{|i(x)|}$ for all $x \in K$. Then

$$\{|n1_K| : n \in \mathbb{N}\} = \{\widehat{|i(n1_K)|} : n \in \mathbb{N}\} = \{\widehat{|n1_{\widehat{K}}|} : n \in \mathbb{N}\}.$$

Therefore the completion of an Archimedean valued field is an Archimedean valued field and the completion of a non-Archimedean valued field is a non-Archimedean valued field. In the latter case, we have that $|K| := \{|x| : x \in K\} = \{\widehat{|x|} : x \in \widehat{K}\} =: \widehat{|\widehat{K}|}$ by Theorem 2.5(12).

The following theorem shows how different a Cauchy complete, non-Archimedean valued field is from its Archimedean analogs $\mathbb{R}$ and $\mathbb{C}$.

Theorem 2.17 ([5, II.1.1]) *Let $(K, |\ \ |)$ be a Cauchy complete non-Archimedean valued field. If $(x_n)_n$ is a sequence of elements of K then*

$$\sum_{n=1}^{\infty} x_n \text{ is convergent in } K \Leftrightarrow \lim_{n\to\infty} x_n = 0.$$

3 Ordered Fields

In this section we will present some examples of ordered fields and we will discuss the concept of Archimedean extension of a field which was introduced by Hans Hahn in 1907 [19].

3.1 Formally Real Fields

By a ring, we will mean a commutative ring with unit $1 \neq 0$. Let A be a ring that is an ordered set such that its additive group $(A, +)$ is an ordered group; that is, it has a total order which is compatible with the addition. The ring A is ordered if for all $x, y \in A$, $x > 0$ and $y > 0$ implies $xy > 0$. Note that an ordered ring is necessarily an integral domain. A field that is an ordered ring will be called an *ordered field*.

Definition 3.1 A field K is *formally real* if it satisfies the following condition: given $n \in \mathbb{N}$ and $a_1, \ldots, a_n \in K$ such that $\sum_{i=1}^{n} a_i^2 = 0$, then $a_1 = \cdots = a_n = 0$.

The following result characterizes the formally real fields as the fields that can be ordered.

Theorem 3.2 ([4, 1.70(5) and 1.71(6)]) *Let K be a field. The following conditions are equivalent.*

(1) *K is formally real,*
(2) *-1 is not a sum of squares in K,*
(3) *There exists an order $\leq$ on K such that $(K, \leq)$ is an ordered field.*

Recall that the characteristic of a field K, denoted by $char(K)$, is the smallest positive integer n such that $\sum_{i=1}^{n} 1_K = 0$ if such a number n exists, and 0 otherwise.

Examples 3.3 (1) If K is a field of non-zero characteristic, then $0 = \sum_{i=1}^{char(K)} 1_K^2$. Hence K is not formally real. Thus if K is formally real then $char(K) = 0$. However, the converse is not true as it can be seen in Example 3.3(4) below.
(2) The field of complex numbers $\mathbb{C}$ cannot be an ordered field since $-1 = i^2$ and therefore it is not formally real.
(3) If K is an ordered field, then we can define an order in the field of formal Laurent series $K((x))$, which is compatible with the addition and multiplication. Thus $K((x))$ can be ordered, and therefore it is formally real. Such an order is defined as follows: for every $z \in K((x))$ there are $r_i \in K$ such that $z = \sum_{i=v}^{\infty} r_i x^i$. We say that $z < 0$ if $z \neq 0$ and $r_v < 0$. Then $z_1 \leq z_2$ if $z_1 = z_2$ or ($z_1 \neq z_2$ and $z_1 - z_2 < 0$).
(4) $\mathbb{Q}_p$ is not formally real because if $p = 2$, then -7 is a square and if $p > 2$ then $1 - p$ is a square ([30, p. 144]). Recall that in a formally real field the squares are non-negative elements. Since $\mathbb{Q} \subset \mathbb{Q}_p$, $char(\mathbb{Q}_p) = 0$.

3.2 General Hahn Fields and the Embedding Theorem

In this subsection we will review the concept of an Archimedean extension of a field as well as the general Hahn fields.

Let K be an ordered field. Two elements $x, y \in K^*$ are *comparable* if there exist $n, m \in \mathbb{N}$ such that $|x|_0 < n|y|_0$ and $|y|_0 < m|x|_0$, where

$$|a|_0 := \max\{a, -a\} = \begin{cases} a, & if\, a \geq 0 \\ -a, & if\, a < 0\,. \end{cases}$$

The relation of being comparable is an equivalence relation on K^* and to denote 'x and y are comparable' we write $x \sim y$. This relation defines a partition of K^* into equivalence classes, which are called the *Archimedean classes of* K. The equivalence class of $x \in K^*$ is denoted by $[x]$; and the set of all the Archimedean classes is denoted by G_K. Then G_K is an ordered abelian group under the order $\prec$ and addition $+$ defined as follows: for every $x, y \in K^*$,

(1) $[x] \prec [y] \iff \forall n \in \mathbb{N}, n|y|_0 < |x|_0 \iff y \not\sim x$ and $|y|_0 < |x|_0$; and
(2) $[x] + [y] := [xy]$.

In this group, the neutral element is $[1_K]$, and $-[x] = [x^{-1}]$ for $x \in K^*$.

Definition 3.4 An ordered field K is *Archimedean* if $G_K = \{[1_K]\}$, that is when any two elements in K^* are comparable.

Theorem 3.5 ([12, 3.7]) *An ordered field K is Archimedean if and only if it satisfies the Archimedean property, i.e. for every $x \in K$, there exists $n \in \mathbb{N}$ such that $|x|_0 < n1_K$.*

Thus for every ordered field K, the group G_K determines the 'Archimedicity' or the 'non-Archimedicity' of K. The field $\mathbb{R}$ of real numbers (the only ordered, Dedekind complete field up to isomorphism) is characterized by the fact that each Archimedean ordered field can be embedded in $\mathbb{R}$ ([20, 3.5]). Hans Hahn in [19] (1907) generalized this property (see Theorem 3.10 below); and by doing so, he ended up with ordered fields that extend all the ordered fields with a given "level of non-Archimedicity".

Definition 3.6 Let E/K be an extension of ordered fields, where the order on E restricted to K coincides with that of K. The field E is an *Archimedean extension of* K if every $x \in E$ is comparable to some $y \in K$. In that case, G_E and G_K are isomorphic ordered groups. An ordered field K is called *Archimedean complete* if it has no proper Archimedean extension fields.

Definition 3.7 Let K be an ordered field. If G is an ordered abelian group isomorphic to G_K, then we say that K *is of type* G and G is called an *Archimedean group of* K.

The simplest Archimedean complete field is $\mathbb{R}$, since it is (up to isomorphism) the only Archimedean complete, ordered field of type $\{0\}$ [12, 3.10]. Archimedean complete fields of other types are given by the general Hahn fields defined in the next result.

Theorem 3.8 ([4, 6.20, 6.21, 7.32], [13, 2.15], [19]) *Let K be a field (not necessarily ordered) and G an ordered abelian group. The set*

$$K((G)) := \{f : G \to K : supp(f) \text{ is well-ordered}\},$$

where $supp(f) := \{x \in G : f(x) \neq 0\}$, is a field under the addition and multiplication defined as follows: for every $f, g \in K((G))$ and $x \in G$,

(1) $(f + g)(x) := f(x) + g(x)$,
(2) $fg(x) := \sum_{a+b=x} f(a)g(b)$.

Fields of the form $K((G))$ are called general Hahn fields.

When K is an ordered field we can define an order on $K((G))$ generalizing the definition of the order in $K((x))$ defined in Example 3.3 (3).

Definition 3.9 (Ordered General Hahn Fields) Let K be an ordered field and consider $\lambda : K((G))^* \to G$, $\lambda(f) = \min\{supp(f)\}$. For $f, g \in K((G))$ we define:

$$f < g \Leftrightarrow f \neq g \text{ and } (g - f)(\lambda(g - f)) > 0.$$

Then $(K((G)), \leq)$ is an ordered field.

The next two results are the main features of the general Hahn fields as ordered fields and mimic the relation between $\mathbb{R}$ and other Archimedean fields.

Theorem 3.10 ([11], [21, 3.1], [4, 1.64], [13, 1.35], [19] (Hahn's Embedding Theorem)) *If K is an ordered field, then for every Archimedean group G of K, there exists an order-preserving field monomorphism σ from K into $\mathbb{R}((G))$ such that $\mathbb{R}((G))$ is an Archimedean extension of $\sigma(K)$.*

Theorem 3.11 ([11, pp. 862–863], [21, 3.2], [19] (Hahn's Completeness Theorem)) *If G is an ordered abelian group then the field $\mathbb{R}((G))$ is (up to isomorphism) the only Archimedean complete, ordered field of type G.*

3.3 Hahn Fields and Levi-Civita Fields

In this subsection we will define a non-Archimedean valuation in some general Hahn fields and the family of the Levi-Civita fields will be introduced.

Definition 3.12 A *Hahn field* is a general Hahn field $K((G))$ for which G is a subgroup of $(\mathbb{R}, +)$ and K is any field.

The distinctive characteristic of a Hahn field among general Hahn fields is that we can define in a natural way a non-Archimedean valuation on the field.

Theorem 3.13 ([33, A.9 pp. 288–292], [34, II.6 corollary, p. 51]) *Let G be a subgroup of $(\mathbb{R}, +)$ and K any field. If the map $|\ \ | : K((G)) \to \mathbb{R}$ is defined by*

$$|f| := \begin{cases} e^{-\min\{supp(f)\}} & if\, f \neq 0 \\ 0 & if\, f = 0, \end{cases}$$

then $(K((G)), |\ \ |)$ is a Cauchy complete non-Archimedean valued field with residue class field isomorphic to K and value group $|K((G))^*| = \{e^g \in \mathbb{R} : g \in G\}$. Moreover, it is spherically complete.

In the following result we introduce the Levi-Civita fields.

Theorem 3.14 ([55, 1.3]) *Let K be any field and let G be a subgroup of $(\mathbb{R}, +)$. Then*

$$L[G, K] := \{f : G \to K \mid supp(f) \cap (-\infty, n] \text{ is finite for every } n \in \mathbb{Z}\}$$

is a subfield of $K((G))$. When we restrict the valuation of $K((G))$ to $L[G, K]$, the latter becomes a Cauchy complete, non-Archimedean valued field with residue class field isomorphic to K and value group $|L[G, K]^*| = \{e^g : g \in G\}$.

Remark 3.15 Fields of the form $L[G, K]$, as defined in Theorem 3.14 above, are called *Levi-Civita fields*. Also, for a given $f \in L[G, K]$, since $supp(f) \cap (-\infty, n]$ is finite for every $n \in \mathbb{Z}$, we say that $supp(f)$ is *left-finite*.

If a field K has a discrete valuation, like $\mathbb{R}((x))$ and $\mathbb{Q}_p$, i.e. when $|K^*|$ is discrete as a subspace of $\mathbb{R}$, then every nonzero element can be written as a limit of a convergent power series ([14, 1.3.5]). The following result shows that in some Levi-Civita fields this also is possible when the valuation is dense, i.e. when $|K^*|$ is dense in $(0, \infty)$.

Lemma 3.16 ([36, Theorem 4.1]) *Let K be a field and let $d : \mathbb{Q} \to K$ be the function defined by*

$$d(x) := \begin{cases} 1 & if\, x = 1 \\ 0 & if\, x \neq 1. \end{cases}$$

Then d is an element of the field $L[\mathbb{Q}, K]$; and for any $r \in \mathbb{Q}$, we have that

$$d^r(x) = \begin{cases} 1 & \text{if } x = r \\ 0 & \text{if } x \neq r. \end{cases}$$

The value group of $(L[\mathbb{Q}, K], |\ \ |)$ is $\{e^{-r} = |d^r| = |d|^r : r \in \mathbb{Q}\}$. Furthermore, every nonzero element f in $L[\mathbb{Q}, K]$ is the sum of a convergent generalized power series with respect to the valuation on $L[\mathbb{Q}, K]$, specifically:

$$f = \sum_{r \in \mathbb{Q}} f(r)d^r = \sum_{r \in supp(f)} f(r)d^r.$$

Additionally, every generalized power series of the form $\sum_{r\in\mathbb{Q}} a_r d^r$ for which $\{r \in \mathbb{Q} : a_r \neq 0\} \cap (-\infty, n]$ is finite for every $n \in \mathbb{Z}$, is convergent in $L[\mathbb{Q}, K]$; and if two such series differ in at least one coefficient then their sums are different.

Theorem 3.17 ([12, 3.19]) *Let K be any field and G a subgroup of $(\mathbb{R}, +)$. Then*

(1) *The fields $K((G))$ and $L[G, K]$ coincide if and only if G is discrete.*
(2) *The field $L[G, K]$ is spherically complete if and only if G is discrete.*
(3) *If K is an ordered field, then $K((G))$ is an Archimedean extension of $L[G, K]$ with respect to the order defined in Definition 3.9. If, in addition, K is Archimedean then both $K((G))$ and $L[G, K]$ are of type G (see Definition 3.7).*

3.4 Real Closed Field Extensions of $\mathbb{R}$

Recall that a field K is *algebraically closed* if every polynomial in $K[x]$ has a root in K. If L/K is a field extension then $a \in L$ is *algebraic* over K if it is the root of a polynomial in $K[x]$. If every element of L is algebraic over K, then L is an *algebraic extension* of K. Also, K is *real closed* if K is formally real and does not admit a proper algebraic extension that is formally real. As the reader can see in the following result, a real closed field is an ordered field very similar to $\mathbb{R}$.

Theorem 3.18 ([22, Chapter XI], [4, 1.71(21),1.71(22)], [10, 5.4.4], [6, Chapter 5, Section 4, Lemma 4.1]) *Let K be a field. The following conditions are equivalent.*

(1) *K is real closed,*
(2) *$x^2 + 1$ is irreducible in K and $K(i)$ is algebraically closed ($i^2 = -1$),*
(3) *K is an ordered field, each positive element of K has a square root and every $p \in K[x]$ of odd degree has a root in K,*
(4) *any sentence in the first-order language of fields is true in K if and only if it is true in $\mathbb{R}$,*

(5) *K is an ordered field and the intermediate value theorem holds for all polynomials over K.*

Therefore, in order to develop a theory of Calculus over ordered fields for which the intermediate value theorem holds, then our base field has to be real closed.

Some general Hahn fields are real closed. In fact, if K is a field and G an ordered abelian group, then $K((G))$ is real closed if and only if K is real closed and G is divisible [4, 6.23 (1)–(2)]. Additionally, the Levi-Civita field $L[\mathbb{Q}, K]$ is real closed if and only if K is real closed. [12, 4.12]. From this and from 3.18(2) we can deduce that the Hahn field $\mathbb{R}((\mathbb{Q}))(i) = \mathbb{C}((\mathbb{Q}))$ and the Levi-Civita field $L[\mathbb{Q}, \mathbb{R}](i) = L[\mathbb{Q}, \mathbb{C}]$ are algebraically closed.

Definition 3.19 Let (x_n) be a sequence of elements in an ordered field K. Then (x_n) is *Cauchy* if for every 0-neighborhood U with respect to the order topology in K, there exists $N \in \mathbb{N}$ such that $x_m - x_n \in U$ for all $m, n \geq N$. The sequence (x_n) is *convergent* to $x \in K$ if for every 0-neighborhood U with respect to the order topology in K, there exists $N \in \mathbb{N}$ such that $x_n - x \in U$ for all $n \geq N$. An ordered field K is said to be *Cauchy complete* if every Cauchy sequence of K is convergent in the order topology.

As the next result states, the Levi-Civita field $L[\mathbb{Q}, \mathbb{R}]$ is the smallest real closed field extension of $\mathbb{R}$ that is Cauchy complete with respect to a non-Archimedean order.

Theorem 3.20 ([36, 3.11]) *Let $K/\mathbb{R}$ be a field extension where K is a Cauchy complete ordered field such that*

(1) *the order in K extends the one in $\mathbb{R}$;*
(2) *there exists $\delta \in K$ such that $0 < \delta < r$ for every $r \in \mathbb{R}^+$ and (δ^n) converges to 0 in the order topology; and*
(3) *K is real closed.*

If d is the function defined in Lemma 3.16 then there exists an order-preserving field monomorphism $\sigma : L[\mathbb{Q}, \mathbb{R}] \to K$ defined by

$$\sigma(f) = \sigma\left(\sum_{q \in supp(f)} f(q) d^q \right) = \sum_{q \in supp(f)} f(q) \delta^q.$$

Note that the order topology on $L[\mathbb{Q}, \mathbb{R}]$ coincides with the topology induced by the valuation [12, 5.7, 5.10]. Using this fact, it is possible to adapt the proof of Theorem 3.20 to prove that the Levi-Civita field $L[\mathbb{Q}, \mathbb{R}]$ equipped with the valuation presented in 3.14 is the smallest non-Archimedean valued field that is a real closed field extension of $\mathbb{R}$ and that is Cauchy complete with respect the valuation. This results is stated in the following corollary.

Corollary 3.21 *Let $K/\mathbb{R}$ be a field extension where K is real closed and Cauchy complete with respect to a non-trivial valuation $\widehat{|\ |}$ such that $\widehat{|x|} = 1$ for all $x \in \mathbb{R}^*$. Then, for every $\delta \in K^*$ such that $\widehat{|\delta|} < 1$, there exists a field monomorphism $\sigma : L[\mathbb{Q}, \mathbb{R}] \to K$ defined by*

$$\sigma(f) = \sigma\left(\sum_{q \in supp(f)} f(q)d^q\right) = \sum_{q \in supp(f)} f(q)\delta^q,$$

satisfying $\widehat{|\sigma(f)|} = |f|^\tau$ for all $f \in L[\mathbb{Q}, \mathbb{R}]$, where $\tau > 0$ is such that $\widehat{|\delta|} = |d|^\tau$.

4 The Levi-Civita Fields $\mathcal{R}$ and $\mathcal{C}$

In the rest of the paper, we will focus on presenting an overview of our research on the Levi-Civita fields $\mathcal{R} := L[\mathbb{Q}, \mathbb{R}]$ and $\mathcal{C} := L[\mathbb{Q}, \mathbb{C}]$. For the further discussion, it is convenient to introduce the following terminology.

Definition 4.1 ($\lambda, \sim, \approx$) For $x \neq 0$ in $\mathcal{R}$ or $\mathcal{C}$, we let $\lambda(x) = \min(supp(x))$, which exists because of the left-finiteness of $supp(x)$; and we let $\lambda(0) = +\infty$. Moreover, we denote the value of x at $q \in \mathbb{Q}$ with brackets like $x[q]$.

Given $x, y \in \mathcal{R}^*$ or $\mathcal{C}^*$, we say $x \sim y$ if $\lambda(x) = \lambda(y)$; and we say $x \approx y$ if $\lambda(x) = \lambda(y)$ and $x[\lambda(x)] = y[\lambda(y)]$.

Note that λ describes orders of magnitude; the relation $\approx$ corresponds to agreement up to infinitely small relative error; while $\sim$ corresponds to agreement of order of magnitude and, in the case of $\mathcal{R}$, it is the same equivalence relation introduced in Sect. 3.2. Moreover, we can isomorphically embed $\mathbb{R}$ and $\mathbb{C}$ in $\mathcal{R}$ and $\mathcal{C}$, respectively, as subfields via the map $E : \mathbb{R}, \mathbb{C} \to \mathcal{R}, \mathcal{C}$ defined by

$$E(x)[q] = \begin{cases} x & \text{if } q = 0 \\ 0 & \text{else} \end{cases} . \tag{4.1}$$

Recall that $\mathcal{R}$ is an ordered subfield of the ordered Hahn field $\mathbb{R}((\mathbb{Q}))$, with its order inherited from that of $\mathbb{R}((\mathbb{Q}))$; see Definition 3.9. Note that, given $a < b$ in $\mathcal{R}$, we define the $\mathcal{R}$-interval $[a, b] = \{x \in \mathcal{R} : a \leq x \leq b\}$, with the obvious adjustments in the definitions of the intervals $[a, b[$, $]a, b]$, and $]a, b[$. Moreover, the embedding E in Eq. (4.1) of $\mathbb{R}$ into $\mathcal{R}$ is compatible with the order.

The order leads to the definition of an ordinary absolute value on $\mathcal{R}$: $|x|_0 = \max\{x, -x\}$ which induces the same topology on $\mathcal{R}$ (called the order topology or valuation topology) as that induced by the ultrametric absolute value (non-Archimedean valuation):

$$|x| = e^{-\lambda(x)},$$

as was shown in [47]. Moreover, two corresponding absolute values are defined on $\mathcal{C}$ in the natural way:

$$|x+iy|_0 = \sqrt{x^2+y^2}; \text{ and } |x+iy| = e^{-\lambda(x+iy)} = \max\{|x|, |y|\}.$$

Thus, $\mathcal{C}$ is topologically isomorphic to $\mathcal{R}^2$ provided with the product topology induced by $|\cdot|_0$ (or $|\cdot|$) in $\mathcal{R}$.

Besides the usual order relations on $\mathcal{R}$, some other notations are convenient.

Definition 4.2 **($\ll, \gg$)** Let $x, y \in \mathcal{R}$ be non-negative. We say x is infinitely smaller than y (and write $x \ll y$) if $nx < y$ for all $n \in \mathbb{N}$; we say x is infinitely larger than y (and write $x \gg y$) if $y \ll x$. If $x \ll 1$, we say x is infinitely small; if $x \gg 1$, we say x is infinitely large. Infinitely small numbers are also called infinitesimals or differentials. Infinitely large numbers are also called infinite. Non-negative numbers that are neither infinitely small nor infinitely large are also called finite.

If d is the Levi-Civita number introduced in Lemma 3.16 then it is easy to check that $d^q \ll 1$ if $q > 0$ and $d^q \gg 1$ if $q < 0$ in $\mathbb{Q}$. Moreover, for all $x \in \mathcal{R}$ (resp. $\mathcal{C}$), the elements of $supp(x)$ can be arranged in ascending order, say $supp(x) = \{q_1, q_2, \ldots\}$ with $q_j < q_{j+1}$ for all j; and x can be written as $x = \sum_{j=1}^{\infty} x[q_j]d^{q_j}$, where the series converges in the valuation topology.

Besides being the smallest ordered non-Archimedean field extension of the real numbers that is both complete in the order topology and real closed, the Levi-Civita field $\mathcal{R}$ is of particular interest because of its practical usefulness. Since the supports of the elements of $\mathcal{R}$ are left-finite, it is possible to represent these numbers on a computer. One such application is the computation of derivatives of real functions representable on a computer [43], where both the accuracy of formula manipulators and the speed of classical numerical methods are achieved.

In the following sections, we present a brief overview of recent research done on $\mathcal{R}$ and $\mathcal{C}$; and we refer the interested reader to the respective papers for a more detailed study of any of the research topics summarized below.

5 Calculus on $\mathcal{R}$ and $\mathcal{R}^n$

The following examples show that functions on a finite interval of $\mathcal{R}$ behave in a way that is different from what we would expect under similar conditions in $\mathbb{R}$.

Example 5.1 Let $f_1 : [0, 1] \to \mathcal{R}$ be given by

$$f_1(x) = \begin{cases} d^{-1} & \text{if } 0 \le x < d \\ d^{-1/\lambda(x)} & \text{if } d \le x \ll 1 \\ 1 & \text{if } x \sim 1. \end{cases}$$

Then f_1 is continuous on $[0, 1]$; but for $d \le x \ll 1$, $f_1(x)$ grows without bound.

Example 5.2 Let $f_2 : [-1, 1] \to \mathcal{R}$ be given by

$$f_2(x) = x - x[0].$$

Then f_2 is continuous on $[-1, 1]$. However, f_2 assumes neither a maximum nor a minimum on $[-1, 1]$. The set $f_2([-1, 1]) = \{y \in \mathcal{R} : \lambda(y) > 0\} = \{y \in \mathcal{R} : |y| < 1\}$ is bounded above by any positive real number and below by any negative real number; but it has neither a least upper bound nor a greatest lower bound.

Example 5.3 Let $f_3 : [0, 1] \to \mathcal{R}$ be given by

$$f_3(x) = \begin{cases} 1 \text{ if } x \sim 1 \\ 0 \text{ if } x \ll 1 \end{cases}.$$

Then f_3 is continuous on $[0, 1]$ and differentiable on $]0, 1[$, with $f_3'(x) = 0$ for all $x \in]0, 1[$. We have that $f_3(0) = 0$ and $f_3(1) = 1$; but $f_3(x) \neq 1/2$ for all $x \in [0, 1]$. Moreover, f_3 is not constant on $[0, 1]$ even though $f_3'(x) = 0$ for all $x \in]0, 1[$.

Example 5.4 Let $f_4 : [-1, 1] \to \mathcal{R}$ be given by

$$f_4(x) = x[0] + \sum_{\nu=1}^{\infty} x_\nu d^{3q_\nu} \text{ when } x = x[0] + \sum_{\nu=1}^{\infty} x_\nu d^{q_\nu}.$$

Then $f_4'(x) = 0$ for all $x \in]-1, 1[$. But f_4 is obviously not constant on $[-1, 1]$.

Remark 5.5 The extension f of f_4 to $\mathcal{R}$, that is $f : \mathcal{R} \to \mathcal{R}$ given by $f(x)[q] = x[q/3]$, is differentiable on all of $\mathcal{R}$ with vanishing derivative everywhere. Moreover, f is an example of a nontrivial order preserving field automorphism on $\mathcal{R}$ [39]; in $\mathbb{R}$ (or any other ordered Archimedean field) the identity map is the only order preserving field automorphism.

Example 5.6 Let $f_5 : [-1, 1] \to \mathcal{R}$ be given by

$$f_5(x) = -f_4(x) + x^4,$$

where f_4 is the function from Example 5.4. Then $f_5'(x) = 4x^3$ for all $x \in]-1, 1[$. Thus, $f_5' > 0$ on $]0, 1[$; but f_5 is not increasing on $]0, 1[$: $f_5\left(d^2\right) > f_5(d)$ even though $d^2 < d$. Also f_5' is strictly increasing and $f_5'' \ge 0$ on $]-1, 1[$; but f_5 is not convex on $]-1, 1[$ since $f_5(d) = -d^3 + d^4 < 0 = f_5(0) + f_5'(0)d$.

Example 5.7 Let $f_6 : [-1, 1] \to \mathcal{R}$ be given by

$$f_6(x) = -\left(f_4(x)\right)^2 + x^8,$$

where f_4 is again the function from Example 5.4. Then f_6 is infinitely often differentiable on $]-1, 1[$ with $f_6^{(j)}(0) = 0$ for $1 \leq j \leq 7$ and $f_6^{(8)}(0) = 8! > 0$. But f_6 has a relative maximum at 0.

The difficulties embodied in the examples above are not specific to $\mathcal{R}$, but are common to all non-Archimedean ordered fields; and they result from the fact that $\mathcal{R}$ is disconnected in the topology induced by the order. This makes developing Analysis on the field more difficult than in the real case; for example, the existence of nonconstant functions whose derivatives vanish everywhere on an interval (as in Example 5.4) makes integration much harder and renders the solutions of the simplest initial value problems (e.g. $y' = 0$; $y(0) = 0$) not unique. To circumvent such difficulties, different approaches have been employed. For example, by imposing stronger conditions on the function than in the real case, we obtain versions of the intermediate value theorem, the inverse function theorem, the mean value theorem and the implicit function theorem [9, 42, 49, 51]; by carefully defining a measure on $\mathcal{R}$ in [40, 46], we succeed in developing an integration theory with similar properties to those of the Lebesgue integral of Real Analysis; and by using a stronger concept of continuity and differentiability than in the real case, one-dimensional and multi-dimensional optimization results similar to those from Real Analysis have been obtained for $\mathcal{R}$-valued functions [52, 53].

5.1 Locally Uniformly Differentiable and Weakly Locally Uniformly Differentiable Functions from $\mathcal{R}$ to $\mathcal{R}$

In [49, 51], we focus our attention on $\mathcal{R}$-valued functions of one variable. We study the properties of locally uniformly differentiable (LUD) functions at a point $x_0 \in \mathcal{R}$ or on an open subset A of $\mathcal{R}$. In particular, we show that LUD functions are C^1, they include all polynomial functions, and they are closed under addition, multiplication and composition. Then we generalize the definition of local uniform differentiability to any order. In particular, we study the properties of LUD^2 functions at a point $x_0 \in \mathcal{R}$ or on an open subset A of $\mathcal{R}$; and we show that LUD^2 functions are C^2, they include all polynomial functions, and they are closed under addition, multiplication and composition. Finally, we formulate and prove an inverse function theorem as well as a local intermediate value theorem and a local mean value theorem for these functions. Here we only recall the main definitions and results (without proofs) and refer the reader to [49, 51] for the details.

Definition 5.8 Let $A \subseteq \mathcal{R}$ be open and let $f : A \to \mathcal{R}$. We say that f is uniformly differentiable (UD) on A if f is differentiable on A and for every $\epsilon > 0$ in $\mathcal{R}$ there exists $\delta > 0$ in $\mathcal{R}$ such that, whenever $x, y \in A$ with $|y - x|_0 < \delta$, we have that $|f(y) - f(x) - f'(x)(y - x)|_0 \leq \epsilon |y - x|_0$.

Definition 5.9 Let $A \subseteq \mathcal{R}$ be open, let $f : A \to \mathcal{R}$, and let $x_0 \in A$ be given. We say that f is locally uniformly differentiable (LUD) at x_0 if there exists a neighborhood Ω of x_0 in A such that f is uniformly differentiable on Ω. Moreover, we say that f is locally uniformly differentiable (LUD) on A if f is LUD at every point in A.

Definition 5.10 Let $A \subset \mathcal{R}$ be open, let $f : A \to \mathcal{R}$, and let $n \in \mathbb{N} \cup \{0\}$ be given. Then we say that f is UD^n on A if f is n times differentiable on A and for every $\epsilon > 0$ in $\mathcal{R}$ there exists $\delta > 0$ in $\mathcal{R}$ such that, whenever $x, y \in A$ with $|y - x|_0 < \delta$, we have that

$$\left| f(y) - \sum_{k=0}^{n} \frac{f^{(k)}(x)}{k!}(y-x)^k \right|_0 \leq \epsilon |y - x|_0^n.$$

Definition 5.11 Let $A \subseteq \mathcal{R}$ be open, let $f : A \to \mathcal{R}$, let $x_0 \in A$, and let $n \in \mathbb{N} \cup \{0\}$ be given. We say that f is LUD^n at x_0 if there exists a neighborhood Ω of x_0 in A such that f is UD^n on Ω. Moreover, we say that f is LUD^n on A if f is LUD^n at every point in A.

Note that, for $n = 0$, UD^n means uniformly continuous; and hence LUD^0 means locally uniformly continuous.

Definition 5.12 Let $A \subseteq \mathcal{R}$ be open, let $f : A \to \mathcal{R}$, and let $x_0 \in A$ be given. We say that f is LUD$^\infty$ at x_0 if f is LUD^n at x_0 for every $n \in \mathbb{N}$. Moreover, we say that f is LUD$^\infty$ on A if f is LUD$^\infty$ at every point in A.

Theorem 5.13 (Inverse Function Theorem) *Let $A \subseteq \mathcal{R}$ be open, let $f : A \to \mathcal{R}$ be LUD on A and let $x_0 \in A$ be such that $f'(x_0) \neq 0$. Then there is a neighborhood Ω of x_0 in A and a function $g : f(\Omega) \to \mathcal{R}$, such that*

(1) *$g = f|_\Omega^{-1}$;*
(2) *$f|_\Omega$ is one-to-one;*
(3) *$f(\Omega)$ is open; and*
(4) *g is LUD on $f(\Omega)$, with $g' = \frac{1}{f' \circ g}$.*

Theorem 5.14 (Local Intermediate Value Theorem) *Let $A \subseteq \mathcal{R}$ be open, let $f : A \to \mathcal{R}$ be LUD on A, and let $x_0 \in A$ be such that $f'(x_0) \neq 0$. Then there is a neighborhood Ω of x_0 such that for any $a < b$ in $f(\Omega)$ and for any $c \in]a, b[$, there is an $x \in \Omega$, strictly between $f^{(-1)}(a)$ and $f^{(-1)}(b)$, such that $f(x) = c$.*

Theorem 5.15 (Local Mean Value Theorem) *Let $A \subseteq \mathcal{R}$ be open, let $f : A \to \mathcal{R}$ be LUD^2 on A, and let $x_0 \in A$ be such that $f''(x_0) \neq 0$. Then there exists a neighborhood Ω of x_0 such that f has the mean value property on Ω. That is, for every $a, b \in \Omega$ with $a < b$, there exists $c \in]a, b[$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

As in the real case, the mean value property can be used to prove other important results. In particular, while L'Hôpital's rule does not hold for differentiable functions on $\mathcal{R}$, we prove the result in [49] under similar conditions to those of the local mean value theorem. To do this we first prove the local equivalent of the Cauchy mean value theorem (Lemma 5.16). The proof is obtained from the mean value property the same way as in the real case.

Lemma 5.16 *Let $A \subseteq \mathcal{R}$ be open, let $f, g : A \to \mathcal{R}$ be LUD^2 on A and let $x_0 \in A$ be such that $f''(x_0) \neq 0$ and $g''(x_0) \neq 0$. Then there exists a neighborhood Ω of x_0 such that for every $a, b \in \Omega$ with $a < b$, there exists $c \in]a, b[$ such that*

$$f'(c)\,(g(b) - g(a)) = g'(c)\,(f(b) - f(a))\,.$$

Theorem 5.17 *Let $A \subseteq \mathcal{R}$ be open, let $f, g : A \to \mathcal{R}$ be LUD^2 on A and let $a \in A$ be such that $f''(a) \neq 0$ and $g''(a) \neq 0$. Furthermore, suppose that $f(a) = g(a) = 0$, that there exists a neighborhood Ω of a in A such that $g'(x) \neq 0$ for every $x \in \Omega \setminus \{a\}$, and that $\lim_{x\to a} f'(x)/g'(x)$ exists. Then*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

In [9], we show that all the results in [49, 51] still hold if we replace local uniform differentiability with the strictly weaker concept of weak local uniform differentiability which we define below.

Definition 5.18 Let $A \subseteq \mathcal{R}$ be open, let $f : A \to \mathcal{R}$, and let $x_0 \in A$ be given. We say that f is weakly locally uniformly differentiable (abbreviated as WLUD) at x_0 if f is differentiable in a neighbourhood Ω of x_0 in A and if for every $\epsilon > 0$ in $\mathcal{R}$ there exists $\delta > 0$ in $\mathcal{R}$ such that for every $x, y \in]x_0 - \delta, x_0 + \delta[\cap\Omega$ we have that $|f(y) - f(x) - f'(x)(y - x)|_0 \leq \epsilon|y - x|_0$. Moreover, we say that f is WLUD on A if f is WLUD at every point in A.

We extend the WLUD concept to higher orders of differentiability and we define WLUD^n analogously to how LUD^n was defined in [49].

Definition 5.19 Let $A \subseteq \mathcal{R}$ be open, let $f : A \to \mathcal{R}$, let $x_0 \in A$, and let $n \in \mathbb{N}$ be given. We say that f is WLUD^n at x_0 if f is n times differentiable in a neighbourhood Ω of x_0 in A and if for every $\epsilon > 0$ in $\mathcal{R}$ there exists $\delta > 0$ in $\mathcal{R}$ such that for every $x, y \in]x_0 - \delta, x_0 + \delta[\cap\Omega$ we have that

$$\left| f(y) - \sum_{k=0}^{n} \frac{f^{(k)}(x)}{k!}(y - x)^k \right|_0 \leq \epsilon\, |y - x|_0^n\,.$$

Moreover, we say that f is WLUD^n on A if f is WLUD^n at every point in A.

Remark 5.20 A close look at Definition 5.19 shows that f is WLUD^0 at x_0 if and only if f is continuous in a neighborhood around x_0.

Finally in [42], we state and prove a Taylor theorem with remainder for WLUD^n functions on $\mathcal{R}$. As in the real case, the proof of the theorem uses the mean value theorem. However, in the non-Archimedean setting, stronger conditions on the function are needed than in the real case.

Theorem 5.21 *(Taylor's Theorem with Remainder) Let $A \subseteq \mathcal{R}$ be open, let $n \in \mathbb{N}$ be given, and let $f : A \to \mathcal{R}$ be WLUD^{n+2} on A. Assume further that $f^{(m)}$ is WLUD^2 on A for $0 \leq m \leq n$. Then, for every $x \in A$, there exists a neighborhood U of x in A such that, for any $y \in U$, there exists $c \in [\min(y, x), \max(y, x)]$ such that*

$$f(y) = \sum_{k=0}^{n} \frac{f^{(k)}(x)}{k!}(y-x)^k + \frac{f^{(n+1)}(c)}{(n+1)!}(y-x)^{n+1}.$$

Then we generalize the concept of weak local uniform differentiability to functions from $\mathcal{R}^n$ to $\mathcal{R}^m$ with $m, n \in \mathbb{N}$. Moreover, we formulate and prove the inverse function theorem for WLUD functions from $\mathcal{R}^n$ to $\mathcal{R}^n$ and the implicit function theorem for WLUD functions from $\mathcal{R}^n$ to $\mathcal{R}^m$ with $m < n$ in $\mathbb{N}$.

5.2 WLUD Functions from $\mathcal{R}^n$ to $\mathcal{R}^m$

Throughout this section, let A denote an open subset of $\mathcal{R}^n$; consequently, whenever we speak of a ball $B_\delta(\boldsymbol{x}) := \{\boldsymbol{y} \in \mathcal{R} : |\boldsymbol{y} - \boldsymbol{x}|_0 < \delta\}$ around a point $\boldsymbol{x}$ in A, it is assumed that $\delta > 0$ is small enough so that $B_\delta(\boldsymbol{x}) \subset A$. We will state the main definitions and results here and we refer the reader to [42] for the details.

Notation 5.22 Let $\boldsymbol{f} : A \to \mathcal{R}^m$ be differentiable at $\boldsymbol{x} \in A$. Then $\boldsymbol{Df}(\boldsymbol{x})$ denotes the linear map from $\mathcal{R}^n$ to $\mathcal{R}^m$ defined by the $m \times n$ Jacobian matrix of $\boldsymbol{f}$ at $\boldsymbol{x}$:

$$\begin{pmatrix} \boldsymbol{f}_1^1(\boldsymbol{x}) & \boldsymbol{f}_2^1(\boldsymbol{x}) & \dots & \boldsymbol{f}_n^1(\boldsymbol{x}) \\ \boldsymbol{f}_1^2(\boldsymbol{x}) & \boldsymbol{f}_2^2(\boldsymbol{x}) & \dots & \boldsymbol{f}_n^2(\boldsymbol{x}) \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{f}_1^m(\boldsymbol{x}) & \boldsymbol{f}_2^m(\boldsymbol{x}) & \dots & \boldsymbol{f}_n^m(\boldsymbol{x}) \end{pmatrix}$$

with $\boldsymbol{f}_j^i(\boldsymbol{x}) = \frac{\partial f_i}{\partial x_j}(\boldsymbol{x})$ for $1 \leq i \leq m$ and $1 \leq j \leq n$.

Definition 5.23 (Uniformly Differentiable) Let $\boldsymbol{f} : A \to \mathcal{R}^m$ be differentiable on A. Then we say that $\boldsymbol{f}$ is uniformly differentiable on A if for all $\epsilon > 0$ in $\mathcal{R}$, there exists $\delta > 0$ in $\mathcal{R}$ such that whenever $\boldsymbol{x}, \boldsymbol{y} \in A$ and $|\boldsymbol{y} - \boldsymbol{x}|_0 < \delta$ we have that $|\boldsymbol{f}(\boldsymbol{y}) - \boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{Df}(\boldsymbol{x})(\boldsymbol{y} - \boldsymbol{x})|_0 \leq \epsilon|\boldsymbol{y} - \boldsymbol{x}|_0$.

Definition 5.24 (Weakly Locally Uniformly Differentiable) Let $A \subset \mathcal{R}^n$ be open, let $\boldsymbol{f} : A \to \mathcal{R}^m$, and let $\boldsymbol{x_0} \in A$ be given. Then we say that $\boldsymbol{f}$ is weakly locally uniformly differentiable (WLUD) at $\boldsymbol{x_0}$ if $\boldsymbol{f}$ is differentiable in a neighborhood Ω of $\boldsymbol{x_0}$ in A and if for every $\epsilon > 0$ in $\mathcal{R}$ there exists $\delta > 0$ in $\mathcal{R}$ such that for all $\boldsymbol{x}, \boldsymbol{y} \in B_\delta(\boldsymbol{x_0}) \cap \Omega$, we have that

$$|\boldsymbol{f}(\boldsymbol{y}) - \boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{Df}(\boldsymbol{x})(\boldsymbol{y}-\boldsymbol{x})|_0 \leq \epsilon |\boldsymbol{y}-\boldsymbol{x}|_0.$$

Moreover, we say that $\boldsymbol{f}$ is WLUD on A if $\boldsymbol{f}$ is WLUD at every point in A.

It is clear from the two definitions above that if $\boldsymbol{f}$ is uniformly differentiable on A then $\boldsymbol{f}$ is WLUD at every point in A and hence $\boldsymbol{f}$ is WLUD on A. Moreover, we show that if $\boldsymbol{f} : A \to \mathcal{R}^m$ is WLUD at $\boldsymbol{x_0} \in A$ (resp. on A) then $\boldsymbol{f}$ is C^1 at $\boldsymbol{x_0}$ (resp. on A). Thus, the class of WLUD functions at $\boldsymbol{x_0}$ (resp. on A) is a subset of the class of C^1 functions at $\boldsymbol{x_0}$ (resp. on A). However, this is still large enough to include all polynomial functions. We also show that any linear combination of WLUD functions at $\boldsymbol{x_0}$ (resp. on A) is again WLUD at $\boldsymbol{x_0}$ (resp. on A). Moreover, we show that if $\boldsymbol{f} : A \to \mathcal{R}^m$ is WLUD at $\boldsymbol{x_0} \in A$ (resp. on A) and if $\boldsymbol{g} : C \to \mathcal{R}^p$ is WLUD at $\boldsymbol{f}(\boldsymbol{x_0})$ (resp. on C), with $\boldsymbol{f}(A) \subseteq C$, then $\boldsymbol{g} \circ \boldsymbol{f}$ is WLUD at $\boldsymbol{x_0}$ (resp. on A).

Theorem 5.25 (Inverse Function Theorem) *Let $\boldsymbol{f} : A \to \mathcal{R}^n$ be WLUD on A and let $\boldsymbol{t_0} \in A$ be such that $J\boldsymbol{f}(\boldsymbol{t_0}) \neq 0$. Then there is a neighborhood Ω of $\boldsymbol{t_0}$ such that:*

(1) *$\boldsymbol{f}|_\Omega$ is one-to-one;*
(2) *$\boldsymbol{f}(\Omega)$ is open;*
(3) *the inverse $\boldsymbol{g}$ of $\boldsymbol{f}|_\Omega$ is WLUD on $\boldsymbol{f}(\Omega)$; and $\boldsymbol{Dg}(\boldsymbol{x}) = [\boldsymbol{Df}(\boldsymbol{t})]^{-1}$ for $\boldsymbol{t} \in \Omega$ and $\boldsymbol{x} = \boldsymbol{f}(\boldsymbol{t})$.*

As in the real case, the inverse function theorem is used to prove the implicit function theorem. Let $A \subseteq \mathcal{R}^n$ be open and let $\boldsymbol{\Phi} : A \to \mathcal{R}^m$ be WLUD on A. For $\boldsymbol{t} = (t_1, \ldots, t_{n-m}, t_{n-m+1}, \ldots, t_n) \in A$, let

$$\hat{\boldsymbol{t}} = (t_1, \ldots, t_{n-m}) \text{ and } \tilde{J}\boldsymbol{\Phi}(\boldsymbol{t}) = \det\left(\frac{\partial(\Phi_1, \ldots, \Phi_m)}{\partial(t_{n-m+1}, \ldots, t_n)}\right).$$

Theorem 5.26 *Let $\boldsymbol{\Phi} : A \to \mathcal{R}^m$ be WLUD on A, where $A \subseteq \mathcal{R}^n$ is open and $1 \leq m < n$. Let $\boldsymbol{t_0} \in A$ be such that $\boldsymbol{\Phi}(\boldsymbol{t_0}) = \boldsymbol{0}$ and $\tilde{J}\boldsymbol{\Phi}(\boldsymbol{t_0}) \neq 0$. Then there exist a neighborhood U of $\boldsymbol{t_0}$, a neighborhood R of $\hat{\boldsymbol{t}}_0$ and $\boldsymbol{\phi} : R \to \mathcal{R}^m$ that is WLUD on R such that*

$$\tilde{J}\boldsymbol{\Phi}(\boldsymbol{t}) \neq 0 \textit{ for all } \boldsymbol{t} \in U,$$

and

$$\{\boldsymbol{t} \in U : \boldsymbol{\Phi}(\boldsymbol{t}) = \boldsymbol{0}\} = \{(\hat{\boldsymbol{t}}, \boldsymbol{\phi}(\hat{\boldsymbol{t}})) : \hat{\boldsymbol{t}} \in R\}.$$

Remark 5.27 All the results in [9, 42, 49, 51] have been proved in a more general context than the Levi-Civita field. More specifically, in those papers, $\mathcal{R}$ is replaced by any non-Archimedean ordered field extension of the real numbers that is real closed and Cauchy complete in the topology induced by the order, which we denote by $\mathcal{N}$.

6 Review of Power Series and Analytic Functions

Power series on the Levi-Civita field $\mathcal{R}$ have been studied in details in [36, 38, 44, 47, 48]; work prior to that had been mostly restricted to power series with real coefficients. In [23–25, 27], they could be studied for infinitely small arguments only, while in [7], using the newly introduced weak topology (see Definition 6.4 below), also finite arguments were possible. Moreover, power series over complete valued fields in general have been studied by Schikhof [33], Alling [4] and others in valuation theory, but always in the valuation topology.

In [44], we study the general case when the coefficients in the power series are Levi-Civita numbers (i.e. elements of $\mathcal{R}$ or $\mathcal{C}$). We study the convergence of sequences and series in both the valuation (order) topology and the weak topology; and we derive convergence criteria for power series in both topologies.

In [47] it is shown that, within their domain of convergence, power series are infinitely often differentiable and the derivatives to any order are obtained by differentiating the power series term by term. Also, power series can be re-expanded around any point in their domain of convergence. We then study a class of functions that are given locally by power series (which we call analytic functions) and show that they are closed under arithmetic operations and compositions and they are infinitely often differentiable with the derivative functions of all orders being analytic themselves.

In [48], we focus on the proof of the intermediate value theorem for analytic functions. Given a function f that is analytic on an interval $[a, b]$ and a value S between $f(a)$ and $f(b)$, we use iteration to construct a sequence of numbers in $[a, b]$ that converges in the valuation topology to a point $c \in [a, b]$ such that $f(c) = S$. The proof is quite involved, making use of many of the results proved in [44, 47] as well as some results from Real Analysis.

Finally, in [38], we state and prove necessary and sufficient conditions for the existence of relative extrema. Then we use that as well as the intermediate value theorem and its proof to prove the extreme value theorem, the mean value theorem, and the inverse function theorem for functions that are analytic on an interval $[a, b]$, thus showing that such functions behave as nicely as real analytic functions.

In the following, we summarize some of the key results in [38, 44, 47, 48]. We start with a brief review of the convergence of sequences in two different topologies.

6.1 Convergence of Sequences in Two Topologies

Definition 6.1 A sequence (s_n) in $\mathcal{R}$ or $\mathcal{C}$ is called regular if the union of the supports of all members of the sequence is a left-finite subset of $\mathbb{Q}$.

Definition 6.2 We say that a sequence (s_n) converges strongly in $\mathcal{R}$ or $\mathcal{C}$ if it converges in the valuation topology.

The fields $\mathcal{R}$ and $\mathcal{C}$ are complete with respect to the valuation topology; and a detailed study of strong convergence can be found in [36, 44].

Since power series with real (complex) coefficients do not converge strongly for any nonzero real (complex) argument, it is advantageous to study a new kind of convergence. We do that by defining a family of semi-norms on $\mathcal{R}$ or $\mathcal{C}$, which induces a topology weaker than the valuation topology and called weak topology [7, 36, 37, 44].

Definition 6.3 Given $r \in \mathbb{R}$, we define a mapping $\|\cdot\|_r : \mathcal{R}$ or $\mathcal{C} \to \mathbb{R}$ as follows: $\|x\|_r = \max\{|x[q]|_0 : q \in \mathbb{Q}$ and $q \le r\}$.

The maximum in Definition 6.3 exists in $\mathbb{R}$ since, for any $r \in \mathbb{R}$, only finitely many of the $x[q]$'s considered do not vanish.

Definition 6.4 A sequence (s_n) in $\mathcal{R}$ (resp. $\mathcal{C}$) is said to be weakly convergent if there exists $s \in \mathcal{R}$ (resp. $\mathcal{C}$), called the weak limit of the sequence (s_n), such that for all $\epsilon > 0$ in $\mathbb{R}$, there exists $N \in \mathbb{N}$ such that $\|s_m - s\|_{1/\epsilon} < \epsilon$ for all $m \ge N$.

It is shown [7] that $\mathcal{R}$ and $\mathcal{C}$ are not Cauchy complete with respect to the weak topology and that strong convergence implies weak convergence to the same limit. A detailed study of weak convergence is found in [7, 36, 37, 44].

6.2 Power Series

In the following, we review strong and weak convergence criteria for power series, Theorems 6.5 and 6.6, the proofs of which are given in [44]. We also note that Theorem 6.5 is a special case of the result on page 59 of [33].

Theorem 6.5 (Strong Convergence Criterion for Power Series) *Let (a_n) be a sequence in $\mathcal{R}$ (resp. $\mathcal{C}$), and let*

$$\lambda_0 = \limsup_{n\to\infty} \left(\frac{-\lambda(a_n)}{n} \right) \text{ in } \mathbb{R} \cup \{-\infty, \infty\}.$$

Let $x_0 \in \mathcal{R}$ (resp. $\mathcal{C}$) be fixed and let $x \in \mathcal{R}$ (resp. $\mathcal{C}$) be given. Then the power series $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ converges strongly if $\lambda(x - x_0) > \lambda_0$ and diverges in the valuation topology if $\lambda(x - x_0) < \lambda_0$ or if $\lambda(x - x_0) = \lambda_0$ and $-\lambda(a_n)/n > \lambda_0$ for infinitely many n.

Theorem 6.6 (Weak Convergence Criterion for Power Series) *Let* (a_n) *be a sequence in* $\mathcal{R}$ *(resp.* $\mathcal{C}$*), and let* $\lambda_0 = \limsup_{n\to\infty}(-\lambda(a_n)/n) \in \mathbb{Q}$*. Let* $x_0 \in \mathcal{R}$ *(resp.* $\mathcal{C}$*) be fixed, and let* $x \in \mathcal{R}$ *(resp.* $\mathcal{C}$*) be such that* $\lambda(x - x_0) = \lambda_0$*. For each* $n \geq 0$*, let* $b_n = a_n d^{n\lambda_0}$*. Suppose that the sequence* (b_n) *is regular and write* $\bigcup_{n=0}^{\infty} supp(b_n) = \{q_1, q_2, \ldots\}$*; with* $q_{j_1} < q_{j_2}$ *if* $j_1 < j_2$*. For each n, write* $b_n = \sum_{j=1}^{\infty} b_{n_j} d^{q_j}$*, where* $b_{n_j} = b_n[q_j]$*. Let*

$$\eta = \frac{1}{\sup\left\{\limsup_{n\to\infty} |b_{n_j}|_0^{1/n} : j \geq 1\right\}} \text{ in } \mathbb{R} \cup \{\infty\}, \tag{6.1}$$

with the conventions $1/0 = \infty$ *and* $1/\infty = 0$*. Then* $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ *converges absolutely in the weak topology if* $|(x - x_0)[\lambda_0]|_0 < \eta$ *and diverges in the weak topology if* $|(x - x_0)[\lambda_0]|_0 > \eta$*.*

Remark 6.7 The number η in Eq. (6.1) is referred to as the radius of weak convergence of the power series $\sum_{n=0}^{\infty} a_n (x - x_0)^n$.

As an immediate consequence of Theorem 6.6, we obtain the following result which allows us to extend real and complex functions representable by power series to the Levi-Civita fields $\mathcal{R}$ and $\mathcal{C}$. This result is of particular interest for the application [43] mentioned in Sect. 4 above and discussed in Sect. 9 below.

Corollary 6.8 (Power Series with Purely Real or Complex Coefficients) *Let* $\sum_{n=0}^{\infty} a_n X^n$ *be a power series with purely real (resp. complex) coefficients and with classical radius of convergence equal to* η*. Let* $x \in \mathcal{R}$ *(resp.* $\mathcal{C}$*), and let* $A_n(x) = \sum_{j=0}^{n} a_j x^j \in \mathcal{R}$ *(resp.* $\mathcal{C}$*). Then, for* $|x|_0 < \eta$ *and* $|x|_0 \not\approx \eta$*, the sequence* $(A_n(x))$ *converges absolutely weakly. We define the limit to be the continuation of the power series to* $\mathcal{R}$ *(resp.* $\mathcal{C}$*).*

Definition 6.9 (The Functions Exp, Cos, Sin, Cosh, and Sinh) By Corollary 6.8, the series

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}, \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}, \text{ and } \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$$

converge absolutely weakly in $\mathcal{R}$ (resp. $\mathcal{C}$) for any $x \in \mathcal{R}$ (resp. $\mathcal{C}$), at most finite in (ordinary) absolute value (that is, for $\lambda(x) \geq 0$). For any such x, define

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!};$$

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!};$$

$$\sin(x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!};$$

$$\cosh(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!};$$

$$\sinh(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}.$$

A detailed study of the transcendental functions introduced on $\mathcal{R}$ and $\mathcal{C}$ in Definition 6.9 can be found in [36]. In particular, we show that addition theorems similar to the real ones hold, which is essential for the implementation of these functions on a computer (see Section 1.5 in [36]).

6.3 Analytic Functions

In this subsection, we review the algebraic and analytical properties of a class of functions that are given locally by power series and we refer the reader to [38, 47, 48] for a more detailed study.

Definition 6.10 Let $a < b$ in $\mathcal{R}$ be given and let $f : [a, b] \to \mathcal{R}$. Then we say that f is analytic on $[a, b]$ if for all $x \in [a, b]$ there exists a positive $\delta \sim b - a$ in $\mathcal{R}$, and there exists a regular sequence $(a_n(x))$ in $\mathcal{R}$ such that, under weak convergence, $f(y) = \sum_{n=0}^{\infty} a_n(x)(y-x)^n$ for all $y \in]x-\delta, x+\delta[\cap [a, b]$.

It is shown in [47] that if f is analytic on $[a, b]$ then f is bounded on $[a, b]$; also, if g is analytic on $[a, b]$ and $\alpha \in \mathcal{R}$ then $f + \alpha g$ and $f \cdot g$ are analytic on $[a, b]$. Moreover, the composition of analytic functions is analytic. Furthermore, using the fact that power series on $\mathcal{R}$ are infinitely often differentiable within their domain of convergence and the derivatives to any order are obtained by differentiating the power series term by term [47], we obtain the following result.

Theorem 6.11 *Let $a < b$ in $\mathcal{R}$ be given, and let $f : [a, b] \to \mathcal{R}$ be analytic on $[a, b]$. Then f is infinitely often differentiable on $[a, b]$, and for any positive integer m, we have that $f^{(m)}$ is analytic on $[a, b]$. Moreover, if f is given locally around $x_0 \in [a, b]$ by $f(x) = \sum_{n=0}^{\infty} a_n(x_0)(x-x_0)^n$, then $f^{(m)}$ is given by*

$$f^{(m)}(x) = \sum_{n=m}^{\infty} n(n-1)\cdots(n-m+1)\, a_n(x_0)(x-x_0)^{n-m}.$$

In particular, we have that $a_m(x_0) = f^{(m)}(x_0)/m!$ for all $m = 0, 1, 2, \ldots$.

In [48], we prove the intermediate value theorem for analytic functions on an interval $[a, b]$.

Theorem 6.12 (Intermediate Value Theorem) *Let $a < b$ in $\mathcal{R}$ be given and let $f : [a, b] \to \mathcal{R}$ be analytic on $[a, b]$. Then f assumes on $[a, b]$ every intermediate value between $f(a)$ and $f(b)$.*

Since Theorem 6.12 is a central result in the study of power series and analytic functions, we present in the following the key steps of the proof and refer the reader to [48] for the detailed (lengthy) proof.

- Without loss of generality, we may assume that f is not constant on $[a, b]$. Let $F : [0, 1] \to \mathcal{R}$ be given by

$$F(x) = f((b-a)x + a) - \frac{f(a) + f(b)}{2}.$$

  Then F is analytic on $[0, 1]$; and f assumes on $[a, b]$ every intermediate value between $f(a)$ and $f(b)$ if and only if F assumes on $[0, 1]$ every intermediate value between $F(0) = (f(a) - f(b))/2$ and $F(1) = (f(b) - f(a))/2 = -F(0)$. So without loss of generality, we may assume that $a = 0$, $b = 1$, and $f = F$. Also, since scaling the function by a constant factor does not affect the existence of intermediate values, we may assume that the index of f, $i(f) := \min\{supp(f(x)) : x \in [0, 1]\}$, is equal to 0.
- We define $f_R : [0, 1] \cap \mathbb{R} \to \mathbb{R}$ by $f_R(X) = f(X)[0]$. Then f_R is a real-valued analytic function on the real interval $[0, 1] \cap \mathbb{R}$. Let S be between $f(a) = f(0)$ and $f(b) = f(1)$; and let $S_R = S[0]$. Then S_R is a real value between $f_R(0)$ and $f_R(1)$. We use the classical intermediate value theorem to find a real point $X_0 \in [0, 1]$ such that $f_R(X_0) = S_R$.
- We use iteration to construct a convergent sequence (x_n) such that $\lambda(x_n) > 0$ and $\lambda(x_{n+2} - x_{n+1}) > \lambda(x_{n+1} - x_n)$ for all $n \in \mathbb{N}$. Let $x = \lim_{n\to\infty} x_n$; then $\lambda(x) > 0$, and we show that $X_0 + x \in [0, 1]$ and $f(X_0 + x) = S$.

A close look at that proof shows that if f is not constant on $[a, b]$ and S is between $f(a)$ and $f(b)$ then there are only finitely many points c in $[a, b]$ such that $f(c) = S$. This is crucial for the proof of the extreme value theorem for the analytic functions in [38].

In [38], we complete the study of analytic functions: we state and prove necessary and sufficient conditions for the existence of relative extrema; then we prove the extreme value theorem, the mean value theorem and the inverse function theorem for these functions, thus showing that analytic functions have all the nice properties of real analytic functions.

Theorem 6.13 *Let $a < b$ in $\mathcal{R}$ be given, let $f : [a, b] \to \mathcal{R}$ be analytic on $[a, b]$, let $x_0 \in]a, b[$, and let $m \in \mathbb{N}$ be the order of the first nonvanishing derivative of f at x_0. Then f has a relative extremum at x_0 if and only if m is even. In that case (m is even), the extremum is a minimum if $f^{(m)}(x_0) > 0$ and a maximum if $f^{(m)}(x_0) < 0$.*

Theorem 6.14 (Extreme Value Theorem) *Let $a < b$ in $\mathcal{R}$ be given and let $f : [a, b] \to \mathcal{R}$ be analytic on $[a, b]$. Then f assumes a maximum and a minimum on $[a, b]$.*

Using the intermediate value theorem and the extreme value theorem, then the following results become easy to prove.

Corollary 6.15 *Let $a < b$ in $\mathcal{R}$ be given and let $f : [a, b] \to \mathcal{R}$ be analytic on $[a, b]$. Then there exist $m, M \in \mathcal{R}$ such that $f([a, b]) = [m, M]$.*

Corollary 6.16 (Mean Value Theorem) *Let $a < b$ in $\mathcal{R}$ be given and let $f : [a, b] \to \mathcal{R}$ be analytic on $[a, b]$. Then there exists $c \in]a, b[$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Corollary 6.17 *Let $a < b$ in $\mathcal{R}$ be given, and let $f : [a, b] \to \mathcal{R}$ be analytic on $[a, b]$. Then the following are true.*

(i) *If $f'(x) \neq 0$ for all $x \in]a, b[$ then either $f'(x) > 0$ for all $x \in]a, b[$ and f is strictly increasing on $[a, b]$, or $f'(x) < 0$ for all $x \in]a, b[$ and f is strictly decreasing on $[a, b]$.*
(ii) *If $f'(x) = 0$ for all $x \in]a, b[$, then f is constant on $[a, b]$.*

Corollary 6.18 (Inverse Function Theorem) *Let $a < b$ in $\mathcal{R}$ be given, let $f : [a, b] \to \mathcal{R}$ be analytic on $[a, b]$, and let $x_0 \in]a, b[$ be such that $f'(x_0) > 0$ (resp. $f'(x_0) < 0$). Then there exists $\delta > 0$ in $\mathcal{R}$ such that*

(i) *$f' > 0$ and f is strictly increasing (resp. $f' < 0$ and f is strictly decreasing) on $[x_0 - \delta, x_0 + \delta]$.*
(ii) *$f([x_0 - \delta, x_0 + \delta]) = [m, M]$ where $m = f(x_0 - \delta)$ and $M = f(x_0 + \delta)$ (resp. $m = f(x_0 + \delta)$ and $M = f(x_0 - \delta)$).*
(iii) *$\exists g : [m, M] \to [x_0 - \delta, x_0 + \delta]$, strictly increasing (resp. strictly decreasing) on $[m, M]$, such that*

  - *g is the inverse of f on $[x_0 - \delta, x_0 + \delta]$;*
  - *g is differentiable on $[m, M]$; and for all $y \in [m, M]$,*

$$g'(y) = \frac{1}{f'(g(y))}.$$

Remark 6.19 Since power series over $\mathcal{R}$ are analytic on any interval within their domain of convergence, all the results of Sect. 6.3 hold as well for power series on any interval in which the series converges weakly.

7 Measure Theory and Integration

Using the nice smoothness properties of power series and analytic functions, summarized above, we develop a Lebesgue-like measure and integration theory on $\mathcal{R}$ in [40, 46] that uses the analytic functions studied in Sect. 6.3 as the building blocks for measurable functions instead of the step functions used in the real case. This was possible in particular because the family $\mathcal{S}(a,b)$ of analytic functions on a given interval $I(a,b) \subset \mathcal{R}$ (where $I(a,b)$ denotes any one of the intervals $[a,b]$, $]a,b]$, $[a,b[$ or $]a,b[$) satisfies the following crucial properties.

(1) $\mathcal{S}(a,b)$ is an algebra that contains the identity function;
(2) for all $f \in \mathcal{S}(a,b)$, f is Lipschitz on $I(a,b)$ and there exists an anti-derivative F of f in $\mathcal{S}(a,b)$, which is unique up to a constant;
(3) for all differentiable $f \in \mathcal{S}(a,b)$, if $f' = 0$ on $]a,b[$ then f is constant on $I(a,b)$; moreover, if $f' \geq 0$ on $]a,b[$ then f is nondecreasing on $I(a,b)$.

Notation 7.1 Let $a < b$ in $\mathcal{R}$ be given. Then by $l(I(a,b))$ we will denote the length of the interval $I(a,b)$, that is

$$l(I(a,b)) = \text{ length of } I(a,b) = b - a.$$

7.1 Measurable Sets

Definition 7.2 Let $A \subset \mathcal{R}$ be given. Then we say that A is measurable if for every $\epsilon > 0$ in $\mathcal{R}$, there exist a sequence of mutually disjoint intervals (I_n) and a sequence of mutually disjoint intervals (J_n) such that $\bigcup_{n=1}^{\infty} I_n \subset A \subset \bigcup_{n=1}^{\infty} J_n$, $\sum_{n=1}^{\infty} l(I_n)$ and $\sum_{n=1}^{\infty} l(J_n)$ converge in $\mathcal{R}$, and $\sum_{n=1}^{\infty} l(J_n) - \sum_{n=1}^{\infty} l(I_n) \leq \epsilon$.

Given a measurable set A, then for every $k \in \mathbb{N}$, we can select a sequence of mutually disjoint intervals $\left(I_n^k\right)$ and a sequence of mutually disjoint intervals $\left(J_n^k\right)$ such that $\sum_{n=1}^{\infty} l\left(I_n^k\right)$ and $\sum_{n=1}^{\infty} l\left(J_n^k\right)$ converge in $\mathcal{R}$ for all k,

$$\bigcup_{n=1}^{\infty} I_n^k \subset \bigcup_{n=1}^{\infty} I_n^{k+1} \subset A \subset \bigcup_{n=1}^{\infty} J_n^{k+1} \subset \bigcup_{n=1}^{\infty} J_n^k \text{ and } \sum_{n=1}^{\infty} l\left(J_n^k\right) - \sum_{n=1}^{\infty} l\left(I_n^k\right) \leq d^k$$

for all $k \in \mathbb{N}$. Since $\mathcal{R}$ is Cauchy-complete in the order topology, it follows that $\lim_{k\to\infty} \sum_{n=1}^{\infty} l\left(I_n^k\right)$ and $\lim_{k\to\infty} \sum_{n=1}^{\infty} l\left(J_n^k\right)$ both exist and they are equal. We call the common value of the limits the measure of A and we denote it by $m(A)$. Thus,

$$m(A) = \lim_{k\to\infty} \sum_{n=1}^{\infty} l\left(I_n^k\right) = \lim_{k\to\infty} \sum_{n=1}^{\infty} l\left(J_n^k\right).$$

Contrary to the real case,

$$\sup\left\{\sum_{n=1}^{\infty} l(I_n) : I_n\text{'s are mutually disjoint intervals and } \bigcup_{n=1}^{\infty} I_n \subset A\right\}$$

and

$$\inf\left\{\sum_{n=1}^{\infty} l(J_n) : J_n\text{'s are mutually disjoint intervals and } A \subset \bigcup_{n=1}^{\infty} J_n\right\}$$

need not exist for a given set $A \subset \mathcal{R}$. However, as shown in [46], if A is measurable then both the supremum and infimum exist and they are equal to $m(A)$. This shows that the definition of measurable sets in Definition 7.2 is a natural generalization of that of the Lebesgue measurable sets of real analysis that corrects for the lack of suprema and infima in non-Archimedean ordered fields.

It follows directly from the definition that $m(A) \geq 0$ for any measurable set $A \subset \mathcal{R}$ and that any interval $I(a, b)$ is measurable with measure $m(I(a, b)) = l(I(a, b)) = b - a$. It also follows that if A is a countable union of mutually disjoint intervals $(I_n(a_n, b_n))$ such that $\sum_{n=1}^{\infty}(b_n - a_n)$ converges then A is measurable with $m(A) = \sum_{n=1}^{\infty}(b_n - a_n)$. Moreover, if $B \subset A \subset \mathcal{R}$ and if A and B are measurable, then $m(B) \leq m(A)$.

In [46] we show that the measure defined on $\mathcal{R}$ above has similar properties to those of the Lebesgue measure on $\mathbb{R}$. For example, we show that any subset of a measurable set of measure 0 is itself measurable and has measure 0. We also show that any countable unions of measurable sets whose measures form a null sequence is measurable and the measure of the union is less than or equal to the sum of the measures of the original sets; moreover, the measure of the union is equal to the sum of the measures of the original sets if the latter are mutually disjoint. Furthermore, we show that any finite intersection of measurable sets is also measurable and that the sum of the measures of two measurable sets is equal to the sum of the measures of their union and intersection.

It is worth noting that the complement of a measurable set in a measurable set need not be measurable. For example, $[0, 1]$ and $[0, 1] \cap \mathbb{Q}$ are both measurable with measures 1 and 0, respectively. However, the complement of $[0, 1] \cap \mathbb{Q}$ in $[0, 1]$ is not measurable. On the other hand, if $B \subset A \subset \mathcal{R}$ and if A, B and $A \setminus B$ are all measurable, then $m(A) = m(B) + m(A \setminus B)$.

The example of $[0, 1] \setminus ([0, 1] \cap \mathbb{Q})$ above shows that the axiom of choice is not needed here to construct a nonmeasurable set, as there are many simple examples of nonmeasurable sets. Indeed, any uncountable real subset of $\mathcal{R}$, like $[0, 1] \cap \mathbb{R}$ for example, is not measurable.

7.2 *Measurable Functions and Integration on $\mathcal{R}$*

We define in [46] a measurable function on a measurable set $A \subset \mathcal{R}$ using Definition 7.2 and analytic functions.

Definition 7.3 Let $A \subset \mathcal{R}$ be a measurable subset of $\mathcal{R}$ and let $f : A \to \mathcal{R}$ be bounded on A. Then we say that f is measurable on A if for all $\epsilon > 0$ in $\mathcal{R}$, there exists a sequence of mutually disjoint intervals (I_n) such that $I_n \subset A$ for all n, $\sum_{n=1}^{\infty} l(I_n)$ converges in $\mathcal{R}$, $m(A) - \sum_{n=1}^{\infty} l(I_n) \leq \epsilon$ and f is analytic on I_n for all n.

In [46], we derive a simple characterization of measurable functions and we show that they form an algebra. Then we show that a measurable function is differentiable almost everywhere and that a function measurable on two measurable subsets of $\mathcal{R}$ is also measurable on their union and intersection.

We define the integral of an analytic function over an interval $I(a, b)$ and we use that to define the integral of a measurable function f over a measurable set A. Before we do that, we recall the following result whose proof can be found in [36].

Proposition 7.4 *Let $a < b$ in $\mathcal{R}$ and let $f : I(a, b) \to \mathcal{R}$ be analytic on $I(a, b)$. Then*

- *f is Lipschitz on $I(a, b)$;*
- *$\lim_{x \to a^+} f(x)$ and $\lim_{x \to b^-} f(x)$ exist;*
- *the function $g : [a, b] \to \mathcal{R}$, given by*

$$g(x) = \begin{cases} f(x) & \text{if } x \in I(a, b) \\ \lim_{\xi \to a^+} f(\xi) & \text{if } x = a \\ \lim_{\xi \to b^-} f(\xi) & \text{if } x = b, \end{cases}$$

extends f to an analytic function on $[a, b]$ when $I(a, b) \subsetneq [a, b]$.

Definition 7.5 Let $a < b$ in $\mathcal{R}$, let $f : I(a, b) \to \mathcal{R}$ be analytic on $I(a, b)$, and let F be an analytic anti-derivative of f on $I(a, b)$. Then the integral of f over $I(a, b)$ is the $\mathcal{R}$ number

$$\int_{I(a,b)} f = \lim_{x \to b^-} F(x) - \lim_{x \to a^+} F(x).$$

The limits in Definition 7.5 account for the case when the interval $I(a, b)$ does not include one or both of the end points; and these limits exist by Proposition 7.4 above.

Now let $A \subset \mathcal{R}$ be measurable, let $f : A \to \mathcal{R}$ be measurable and let M be a bound for $|f|_0$ on A. Then for every $k \in \mathbb{N}$, there exists a sequence of mutually disjoint intervals $\left(I_n^k\right)_{n=1}^{\infty}$ such that $\bigcup_{n=1}^{\infty} I_n^k \subset A$, $\sum_{n=1}^{\infty} l\left(I_n^k\right)$ converges, $m(A) - \sum_{n=1}^{\infty} l\left(I_n^k\right) \leq d^k$, and f is analytic on I_n^k for all $n \in \mathbb{N}$. Without loss of generality, we may assume that $I_n^k \subset I_n^{k+1}$ for all $n \in \mathbb{N}$ and for all $k \in \mathbb{N}$. Since $\lim_{n\to\infty} l\left(I_n^k\right) = 0$, and since $\left|\int_{I_n^k} f\right|_0 \leq Ml\left(I_n^k\right)$ (proved in [46] for analytic functions), it follows that

$$\lim_{n\to\infty} \int_{I_n^k} f = 0 \text{ for all } k \in \mathbb{N}.$$

Thus, $\sum_{n=1}^{\infty} \int_{I_n^k} f$ converges in $\mathcal{R}$ for all $k \in \mathbb{N}$ [44].

We show that the sequence $\left(\sum_{n=1}^{\infty} \int_{I_n^k} f\right)_{k=1}^{\infty}$ converges in $\mathcal{R}$; and we define the unique limit as the integral of f over A.

Definition 7.6 Let $A \subset \mathcal{R}$ be measurable and let $f : A \to \mathcal{R}$ be measurable. Then the integral of f over A, denoted by $\int_A f$, is given by

$$\int_A f = \lim_{\substack{\sum_{n=1}^{\infty} l(I_n) \to m(A) \\ \bigcup_{n=1}^{\infty} I_n \subset A \\ I_n' \text{s are mutually disjoint} \\ f \text{ is analytic on } I_n \ \forall\, n}} \sum_{n=1}^{\infty} \int_{I_n} f.$$

It turns out that the integral in Definition 7.6 satisfies similar properties to those of the Lebesgue integral on $\mathbb{R}$ [46]. In particular, we prove the linearity property of the integral and that if $|f|_0 \leq M$ on A then $\left|\int_A f\right|_0 \leq Mm(A)$, where $m(A)$ is the measure of A. We also show that the sum of the integrals of a measurable function over two measurable sets is equal to the sum of its integrals over the union and the intersection of the two sets.

In [40], which is a continuation of the work done in [46] and complements it, we show, among other results, that the uniform limit of a sequence of convergent power series on an interval $I(a, b)$ is again a power series that converges on $I(a, b)$. Then we use that to prove the uniform convergence theorem in $\mathcal{R}$.

Theorem 7.7 *Let $A \subset \mathcal{R}$ be measurable, let $f : A \to \mathcal{R}$, for each $k \in \mathbb{N}$ let $f_k : A \to \mathcal{R}$ be measurable on A, and let the sequence (f_k) converge uniformly to f on A. Then f is measurable on A, $\lim_{k\to\infty} \int_A f_k$ exists, and*

$$\lim_{k\to\infty}\int_A f_k = \int_A f.$$

7.3 Integration on $\mathcal{R}^2$ and $\mathcal{R}^3$

In [50] we generalize the results of [40, 46] to two and three dimensions. In particular, we define a Lebesgue-like measure on $\mathcal{R}^2$ (resp. $\mathcal{R}^3$). Then we define measurable functions on measurable sets using analytic functions in two (resp. three) variables and show how to integrate those measurable functions using iterated integration. The resulting double (resp. triple) integral satisfies similar properties to those of the single integral in [40, 46] as well as those properties satisfied by the double and triple integrals of real calculus. In order to have basic regions, like disks for example, measurable, it turns out that the so-called simple regions defined below, rather than rectangles, are the best choice for the building blocks for measurable sets. We recall the following definitions from [50] which will be needed later in this paper.

Definition 7.8 (Simple Region) Let $G \subset \mathcal{R}^2$. Then we say that G is a simple region if there exist $a \leqslant b$ in $\mathcal{R}$ and analytic functions $h_1, h_2 : I(a,b) \to \mathcal{R}$, with $h_1 \le h_2$ on $I(a,b)$ such that

$$G = \{(x,y) \in \mathcal{R}^2 : y \in I(h_1(x), h_2(x)), x \in I(a,b)\}$$

or

$$G = \{(x,y) \in \mathcal{R}^2 : x \in I(h_1(y), h_2(y)), y \in I(a,b)\}.$$

Definition 7.9 (λ_x and λ_y of a simple region) Let $A \subset \mathcal{R}^2$ be a simple region. If $A = \{(x,y) \in \mathcal{R}^2 : y \in I(h_1(x), h_2(x)), x \in I(a,b)\}$ we define $\lambda_x(A) = \lambda(b-a)$ and $\lambda_y(A) = i(h_2 - h_1)$ on $I(a,b)$ where $i(h_2 - h_1)$ is the index of the analytic function $h_2 - h_1$ on $I(a,b)$: $i(h_2 - h_1) = \min\{\lambda(h_2(x) - h_1(x)) : x \in I(a,b)\}$.

On the other hand, if $A = \{(x,y) \in \mathcal{R}^2 : x \in I(h_1(y), h_2(y)), y \in I(a,b)\}$, we define $\lambda_y(A) = \lambda(b-a)$ and $\lambda_x(A) = i(h_2 - h_1)$ on $I(a,b)$.

If $\lambda_x(A) = \lambda_y(A) = 0$ then we say that A is finite.

Definition 7.10 (Analytic Functions on $\mathcal{R}^2$) Let $A \subset \mathcal{R}^2$ be a simple region. Then we say that $f : A \to \mathcal{R}^2$ is an analytic function on A if, for every $(x_0, y_0) \in A$, there exist a simple region A_0 containing (x_0, y_0) that satisfies $\lambda_x(A_0) = \lambda_x(A)$ and $\lambda_y(A_0) = \lambda_y(A)$, and a regular sequence $(a_{ij})_{i,j=0}^{\infty}$ such that for every $s, t \in \mathcal{R}$, if $(x_0 + s, y_0 + t) \in A \cap A_0$ then

$$f(x_0+s, y_0+t) = \sum_{i,j=0}^{\infty} a_{ij} s^i t^j = f(x_0, y_0) + \sum_{\substack{i,j=0 \\ i+j\neq 0}}^{\infty} a_{ij} s^i t^j,$$

where the power series converges in the weak topology.

Given a simple region $S \subset \mathcal{R}^2$ and an analytic function $f : S \to \mathcal{R}$, we define the index of f on S by

$$i(f) = \min\{\lambda(f(x,y))|(x,y) \in S\},$$

which is shown to exist [50]. We note that $\lambda(f(x,y)) = i(f)$ for almost every $(x,y) \in S \cap (d^{\lambda_x(S)}\mathbb{R} \times d^{\lambda_y(S)}\mathbb{R})$ and for any such point $(x,y) \in S \cap (d^{\lambda_x(S)}\mathbb{R} \times d^{\lambda_y(S)}\mathbb{R})$, we have that $\lambda(f(x',y')) = i(f)$ for all $(x',y') \in S$ satisfying $|x'-x|_0 \ll d^{\lambda_x(S)}$ and $|y'-y|_0 \ll d^{\lambda_y(S)}$.

With the above definitions, we can proceed to define measurable sets, measurable functions, and integration just as we did in $\mathcal{R}$, replacing intervals by simple regions. We can then extend the measure theory and integration to $\mathcal{R}^3$, $\mathcal{R}^4$, etc. in an inductive way and obtain similar properties for the resulting integrals as those for the single integral defined above.

As an application of the integration theory, we develop in [17] a theory of integrable delta functions on the Levi-Civita field $\mathcal{R}$ as well as on $\mathcal{R}^2$ and $\mathcal{R}^3$ with similar properties to the one-dimensional, two-dimensional and three-dimensional Dirac delta functions and which reduce to them when restricted to $\mathbb{R}$, $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively; and we show how those delta functions can be used to solve differential equations that arise in Physics and Engineering.

7.4 Integrable Delta Functions

In various branches of physics, one encounters sources which are nearly instantaneous (if time is the independent variable) or almost localized (if the independent variable is a space coordinate). To avoid the cumbersome studies of the detailed functional dependencies of such sources, one would like to replace them with idealized sources that are truly instantaneous or localized. Typical examples of such sources are the concentrated forces and moments in solid mechanics, the point masses in the theory of the gravitational potential, and the point charges in electrostatics. The field of real numbers $\mathbb{R}$ does not permit a direct representation of the (improper) delta functions used for the description of impulsive (instantaneous) or concentrated (localized) sources. Of course, within the framework of distributions, these concepts can be accounted for in a rigorous fashion, but at the expense of the intuitive interpretation.

The existence of infinitely small numbers and infinitely large numbers in the non-Archimedean Levi-Civita field $\mathcal{R}$ allows us to have well-behaved delta functions.

For example, the function $\delta : \mathcal{R} \to \mathcal{R}$, given by

$$\delta(x) = \begin{cases} \frac{3}{4}d^{-3}\left(d^2 - x^2\right) & \text{if } |x|_0 < d \\ 0 & \text{otherwise,} \end{cases}$$

where d is the positive infinitely small number introduced in Lemma 3.16, is a (one-dimensional) continuous (and piece-wise infinitely differentiable) delta function; it assumes an infinitely large value ($3/4d^{-1}$) at 0, it vanishes at all other real points and its integral on any interval containing $]-d, d[$ is equal to one. In the following we summarize key properties of $\delta(x)$ which remind us of the corresponding properties of the Dirac delta function.

- If $I \subset \mathcal{R}$ is an interval that contains $]-d, d[$ then

$$\int_{x \in I} \delta(x) = 1.$$

  Moreover, if $]-d, d[\cap I = \emptyset$ then

$$\int_{x \in I} \delta(x) = 0.$$

- If $\alpha \neq 0$ in $\mathcal{R}$ and if $I \subset \mathcal{R}$ is an interval containing $\left]-\frac{d}{|\alpha|}, \frac{d}{|\alpha|}\right[$ then

$$\int_{x \in I} \delta(\alpha x) = \frac{1}{|\alpha|}.$$

- Let $I \subset \mathcal{R}$ be an interval containing $]-d, d[$. Then the function $H : I \to \mathcal{R}$, given by

$$H(x) = \begin{cases} 0 & \text{if } x \leqslant -d \\ \frac{3}{4}d^{-3}(d^2 x - \frac{1}{3}x^3) + \frac{1}{2} & \text{if } -d < x < d \\ 1 & \text{if } x \geqslant d \end{cases},$$

  is a measurable anti-derivative of $\delta(x)$ on I that is equal to the Heaviside function on $I \cap \mathbb{R}$.

- If $a < b$ in $\mathcal{R}$ is such that $\lambda(b-a) < 1$ and if $f : I(a, b) \to \mathcal{R}$ is analytic on $I(a, b)$ with $i(f) = 0$ then for any $x_0 \in [a+d, b-d]$, we have that

$$\int_{x \in I(a,b)} f(x)\delta(x - x_0) =_0 f(x_0).$$

8 Optimization

In [52], we consider unconstrained one-dimensional optimization on $\mathcal{R}$. We study general optimization questions and derive first and second order necessary and sufficient conditions for the existence of local maxima and minima of a function on a convex subset of $\mathcal{R}$. We show that for first order optimization, the results are similar to the corresponding real ones. However, for second and higher order optimization, we show that conventional differentiability is not strong enough to just extend the real-case results (see Examples 5.6 and 5.7); and a stronger concept of differentiability, the so-called derivate differentiability (see Definition 8.4 below), is used to solve that difficulty. We also characterize convex functions on convex sets of $\mathcal{R}$ in terms of first and second order derivatives.

8.1 One-Dimensional Optimization

In the following, we review the definitions of derivate continuity and differentiability in one dimension, as well as some related results and we refer the interested reader to [36, 41] for a more detailed study. As before, throughout this section, $I(a, b)$ will denote any one of the intervals $]a, b[$, $]a, b]$, $[a, b[$ or $[a, b]$.

Definition 8.1 Let $a < b$ be given in $\mathcal{R}$ and let $f : I(a, b) \to \mathcal{R}$. Then we say that f is derivate continuous on $I(a, b)$ if there exists $M \in \mathcal{R}$, called a Lipschitz constant of f on $I(a, b)$, such that

$$\left|\frac{f(y) - f(x)}{y - x}\right|_0 \leq M \text{ for all } x \neq y \text{ in } I(a, b).$$

It follows immediately from Definition 8.1 that if $f : I(a, b) \to \mathcal{R}$ is derivate continuous on $I(a, b)$ then f is uniformly continuous and bounded on $I(a, b)$.

Remark 8.2 It is clear that the concept of derivate continuity in Definition 8.1 coincides with that of Lipschitz continuity when restricted to $\mathbb{R}$. We chose to call it derivate continuity here so that, after having defined derivate differentiability in Definition 8.4 and higher order derivate differentiability in Definition 8.6, we can think of derivate continuity as derivate differentiability of “order zero", just as is the case for continuity in $\mathbb{R}$.

Remark 8.3 Definition 8.1 can be generalized in the obvious way to functions on any countable unions of intervals of $\mathcal{R}$.

Definition 8.4 Let $a < b$ be given in $\mathcal{R}$, let $f : I(a, b) \to \mathcal{R}$ be derivate continuous on $I(a, b)$, and let I_d denote the identity function on $I(a, b)$. Then we say that f is derivate differentiable on $I(a, b)$ if for all $x \in I(a, b)$, the function $\frac{f - f(x)}{I_d - x} : I(a, b) \setminus \{x\} \to \mathcal{R}$ is derivate continuous on $I(a, b) \setminus \{x\}$. In this case the unique continuation of $\frac{f - f(x)}{I_d - x}$ to $I(a, b)$ will be called the first derivate function (or simply the derivate function) of f at x and will be denoted by $F_{1,x}$; moreover, the function value $F_{1,x}(x)$ will be called the derivative of f at x and will be denoted by $f'(x)$.

It follows immediately from Definition 8.4 that if $f : I(a,b) \to \mathcal{R}$ is derivate differentiable then f is differentiable in the conventional sense; moreover, the two derivatives at any given point of $I(a,b)$ agree. As for derivate continuity, the definition of derivate differentiability can be generalized to functions on countable unions of intervals of $\mathcal{R}$.

The following result provides a useful tool for checking the derivate differentiability of functions.

Theorem 8.5 *Let $a < b$ be given in $\mathcal{R}$ and let $f : I(a,b) \to \mathcal{R}$ be derivate continuous on $I(a,b)$. Suppose there exists $M \in \mathcal{R}$ and there exists a function $g : I(a,b) \to \mathcal{R}$ such that*

$$\left| \frac{f(y) - f(x)}{y - x} - g(x) \right|_0 \leq M |y - x|_0 \text{ for all } y \neq x \text{ in } I(a,b).$$

Then f is derivate differentiable on $I(a,b)$, with derivative $f' = g$.

Definition 8.6 (n-times Derivate Differentiability) Let $a < b$ be given in $\mathcal{R}$, let $f : I(a,b) \to \mathcal{R}$, and let $n \geq 2$ be given in $\mathbb{N}$. Then we define n-times derivate differentiability of f on $I(a,b)$ inductively as follows: Having defined $(n-1)$-times derivate differentiability, we say that f is n-times derivate differentiable on $I(a,b)$ if f is $(n-1)$-times derivate differentiable on $I(a,b)$ and for all $x \in I(a,b)$, the $(n-1)$st derivate function $F_{n-1,x}$ is derivate differentiable on $I(a,b)$. For all $x \in I(a,b)$, the derivate function $F_{n,x}$ of $F_{n-1,x}$ at x will be called the nth derivate function of f at x, and the number $f^{(n)}(x) = n!F'_{n-1,x}(x)$ will be called the nth derivative of f at x and denoted by $f^{(n)}(x)$.

One of the most useful consequences of the derivate differentiability concept is that it gives rise to a Taylor theorem with remainder while the conventional (topological) differentiability does not. We only state the result here and refer the reader to [36, 41] for its proof. We also note that, as an immediate result of Theorem 8.7, we obtain local expandability in Taylor series around $x_0 \in I(a,b)$ of a given function that is infinitely often derivate differentiable on $I(a,b)$ [36, 41].

Theorem 8.7 (Taylor's Theorem with Remainder) *Let $a < b$ be given in $\mathcal{R}$ and let $f : I(a,b) \to \mathcal{R}$ be n-times derivate differentiable on $I(a,b)$. Let $x \in I(a,b)$ be given, let $F_{n,x}$ be the nth order derivate function of f at x, and let $M_{n,x}$ be a Lipschitz constant of $F_{n,x}$. Then for all $y \in I(a,b)$, we have that*

$$f(y) = f(x) + \sum_{j=1}^{n} \frac{f^{(j)}(x)}{j!} (y-x)^j + r_n(x,y)(y-x)^{n+1},$$

with $\lambda(r_n(x,y)) \geq \lambda\left(M_{n,x}\right)$.

Using Theorem 8.7, we are able to generalize in [52] most of the one-dimensional optimization results of Real Analysis. For example, we obtain the following two results which state necessary and sufficient conditions for the existence of local (relative) extrema.

Theorem 8.8 (Necessary Conditions for Existence of Local Extrema) *Let $a < b$ be given in $\mathcal{R}$, let $m \geq 2$, and let $f : I(a, b) \to \mathcal{R}$ be m-times derivate differentiable on $I(a, b)$. Suppose that f has a local extremum at $x_0 \in]a, b[$ and $l \leq m$ is the order of the first nonvanishing derivative of f at x_0. Then l is even. Moreover, $f^{(l)}(x_0)$ is positive if the extremum is a minimum and negative if the extremum is a maximum.*

Theorem 8.9 (Sufficient Conditions for Existence of Local Extrema) *Let $a < b$ be given in $\mathcal{R}$, let $k \in \mathbb{N}$, and let $f : I(a, b) \to \mathcal{R}$ be $2k$-times derivate differentiable on $I(a, b)$. Let $x_0 \in]a, b[$ be such that $f^{(j)}(x_0) = 0$ for all $j \in \{1, \ldots, 2k - 1\}$ and $f^{(2k)}(x_0) \neq 0$. Then f has a local minimum at x_0 if $f^{(2k)}(x_0) > 0$ and a local maximum if $f^{(2k)}(x_0) < 0$.*

8.2 *Multidimensional Constrained Optimization*

In [41, 53], we generalize the concepts of derivate continuity and differentiability to higher dimensions; and this yields a Taylor theorem with a bounded remainder term for C^m functions (in the derivate sense) from an open subset of $\mathcal{R}^n$ to $\mathcal{R}$.

Definition 8.10 Let $D \subset \mathcal{R}^n$ be open, let $f : D \to \mathcal{R}$, and let $\boldsymbol{u} = \{u_1, u_2, \ldots, u_n\}$ be a unit vector (that is, $|\boldsymbol{u}|_0 = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2} = 1$). For each $\boldsymbol{x} \in D$, let

$$D_{\boldsymbol{x},\boldsymbol{u}} = \{t \in \mathcal{R} : \boldsymbol{x} + t\boldsymbol{u} \in D\}$$

and define $\phi_{\boldsymbol{x},\boldsymbol{u}} : D_{\boldsymbol{x},\boldsymbol{u}} \to \mathcal{R}$ by

$$\phi_{\boldsymbol{x},\boldsymbol{u}}(t) = f(\boldsymbol{x} + t\boldsymbol{u}).$$

Then we say that f is derivate differentiable on D in the direction of $\boldsymbol{u}$ if $\phi_{\boldsymbol{x},\boldsymbol{u}}$ is derivate differentiable on $D_{\boldsymbol{x},\boldsymbol{u}}$ for all $\boldsymbol{x}$ in D. Moreover, the derivative $\phi'_{\boldsymbol{x},\boldsymbol{u}}(0)$ will be called the directional derivative of f at $\boldsymbol{x}$ in the $\boldsymbol{u}$ direction.

Definition 8.11 (Partial Derivatives) Let $D \subset \mathcal{R}^n$ be open, let $f : D \to \mathcal{R}$ and let $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n\}$ denote the standard basis of $\mathcal{R}^n$. Then the partial derivatives of f are defined as the directional derivatives of f in the directions $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$, if these exist. The gradient of f, denoted by ∇f, is defined to be the row vector whose components are the (first order) partial derivatives of f.

Definition 8.12 Let $D \subset \mathcal{R}^n$ be open, let $f : D \to \mathcal{R}$ and let $q \in \mathbb{N}$ be given. Then we say that f is C^q on D if all the partial derivatives of order smaller than or equal to q exist and are derivate continuous on D.

Theorem 8.13 (Taylor's Theorem with Remainder for Functions of Several Variables) *Let $D \subset \mathcal{R}^n$ be open, let $\boldsymbol{x}_0 \in D$ be given and let $f : D \to \mathcal{R}$ be C^q on D. Then there exist $M, \delta > 0$ in $\mathcal{R}$ such that $B_\delta(\boldsymbol{x}_0) := \{\boldsymbol{x} \in \mathcal{R}^n : |\boldsymbol{x} - \boldsymbol{x}_0|_0 < \delta\} \subset D$ and, for all $\boldsymbol{x} \in B_\delta(\boldsymbol{x}_0)$, we have that*

$$f(\boldsymbol{x}) = f(\boldsymbol{x}_0) + \sum_{j=1}^{q} \left(\frac{1}{j!} \sum_{l_1,\ldots,l_j=1}^{n} \left(\partial_{l_1} \cdots \partial_{l_j} f(\boldsymbol{x}_0) \pi_{k=1}^{j} \left(x_{l_k} - x_{0,l_k} \right) \right) \right)$$
$$+ R_{q+1}(\boldsymbol{x}_0, \boldsymbol{x}),$$

where $\left| R_{q+1}(\boldsymbol{x}_0, \boldsymbol{x}) \right|_0 \leq M |\boldsymbol{x} - \boldsymbol{x}_0|_0^{q+1}$.

Then we use Theorem 8.13 to derive necessary and sufficient conditions of second order for the existence of a minimum of an $\mathcal{R}$-valued function on $\mathcal{R}^n$ subject to equality and inequality constraints. More specifically, we solve the problem of minimizing a function $f : \mathcal{R}^n \to \mathcal{R}$, subject to the following set of constraints:

$$\begin{cases} h_1(\boldsymbol{x}) = 0 \\ \quad\vdots \\ h_m(\boldsymbol{x}) = 0 \end{cases} \text{and} \begin{cases} g_1(\boldsymbol{x}) \leq 0 \\ \quad\vdots \\ g_p(\boldsymbol{x}) \leq 0 \end{cases}, \tag{8.1}$$

where all the functions in (8.1) are from $\mathcal{R}^n$ to $\mathcal{R}$. A point $\boldsymbol{x}_0 \in \mathcal{R}^n$ is said to be a feasible point if it satisfies the constraints in (8.1).

Definition 8.14 Let $\boldsymbol{x}_0$ be a feasible point for the constraints in (8.1) and let $I(\boldsymbol{x}_0) = \{l \in \{1, \ldots, p\} : g_l(\boldsymbol{x}_0) = 0\}$. Then we say that $\boldsymbol{x}_0$ is regular for the constraints if $\left\{\nabla h_j(\boldsymbol{x}_0) : j = 1, \ldots, m; \nabla g_l(\boldsymbol{x}_0) : l \in I(\boldsymbol{x}_0)\right\}$ forms a linearly independent subset of vectors in $\mathcal{R}^n$.

The following theorem provides *necessary conditions* of second order for a local minimizer $\boldsymbol{x}_0$ of a function f subject to the constraints in (8.1). The result is a generalization of the corresponding real result [15, 26] and the proof (see [53]) is similar to that of the latter; but one essential difference is the form of the remainder formula in Taylor's theorem. In the real case, the remainder term is related to the second derivative at some intermediate point, while here that is not the case. However, the concept of derivate differentiability puts a bound on the remainder term; and this proves to be instrumental in the proof of the theorem in the non-Archimedean setting.

Theorem 8.15 *Suppose that f, $\{h_j\}_{j=1}^m$, $\{g_l\}_{l=1}^p$ are C^2 on some open set $D \subset \mathcal{R}^n$ containing the point $\boldsymbol{x}_0$ and that $\boldsymbol{x}_0$ is a regular point for the constraints in*

(8.1). If $\boldsymbol{x}_0$ is a local minimizer for f under the given constraints, then there exist $\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_p \in \mathcal{R}$ such that

(i) $\beta_l \geq 0$ *for all* $l \in \{1, \ldots, p\}$,
(ii) $\beta_l g_l(\boldsymbol{x}_0) = 0$ *for all* $l \in \{1, \ldots, p\}$,
(iii) $\nabla f(\boldsymbol{x}_0) + \sum_{j=1}^{m} \alpha_j \nabla h_j(\boldsymbol{x}_0) + \sum_{l=1}^{p} \beta_l \nabla g_l(\boldsymbol{x}_0) = \mathbf{0}$, *and*
(iv) $\boldsymbol{y}^T \left(\nabla^2 f(\boldsymbol{x}_0) + \sum_{j=1}^{m} \alpha_j \nabla^2 h_j(\boldsymbol{x}_0) + \sum_{l=1}^{p} \beta_l \nabla^2 g_l(\boldsymbol{x}_0)\right) \boldsymbol{y} \geq 0$ *for all* $\boldsymbol{y} \in \mathcal{R}^n$ *satisfying* $\nabla h_j(\boldsymbol{x}_0)\boldsymbol{y} = 0$ *for all* $j \in \{1, \ldots, m\}$, $\nabla g_l(\boldsymbol{x}_0)\boldsymbol{y} = 0$ *for all* $l \in L = \{k \in I(\boldsymbol{x}_0) : \beta_k > 0\}$ *and* $\nabla g_l(\boldsymbol{x}_0)\boldsymbol{y} \leq 0$ *for all* $l \in I(\boldsymbol{x}_0) \setminus L$.

In the following theorem, we present second order *sufficient conditions* for a feasible point $\boldsymbol{x}_0$ to be a local minimum of a function f subject to the constraints in (8.1). It is a generalization of the real result [15] and reduces to it, when restricted to functions from $\mathbb{R}^n$ to $\mathbb{R}$. In fact, since ϵ in condition (iv) below is allowed to be infinitely small, the condition $|\nabla h_j(\boldsymbol{x}_0)\boldsymbol{y}|_0 < \epsilon$ would reduce to $\nabla h_j(\boldsymbol{x}_0)\boldsymbol{y} = 0$, when restricted to $\mathbb{R}$. Similarly, one can readily see that the other conditions are mere generalizations of the corresponding real ones. However, the proof (see [53]) is different from that of the real result since the supremum principle does not hold in $\mathcal{R}$.

Theorem 8.16 *Suppose that f, $\{h_j\}_{j=1}^{m}$, $\{g_l\}_{l=1}^{p}$ are C^2 on some open set $D \subset \mathcal{R}^n$ containing the point $\boldsymbol{x}_0$ and that $\boldsymbol{x}_0$ is a feasible point for the constraints in (8.1) such that, for some $\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_p \in \mathcal{R}$ and for some $\epsilon, \gamma > 0$ in $\mathcal{R}$, we have that*

(i) $\beta_l \geq 0$ *for all* $l \in \{1, \ldots, p\}$,
(ii) $\beta_l g_l(\boldsymbol{x}_0) = 0$ *for all* $l \in \{1, \ldots, p\}$,
(iii) $\nabla f(\boldsymbol{x}_0) + \sum_{j=1}^{m} \alpha_j \nabla h_j(\boldsymbol{x}_0) + \sum_{l=1}^{p} \beta_l \nabla g_l(\boldsymbol{x}_0) = \mathbf{0}$, *and*
(iv) $\boldsymbol{y}^T \left(\nabla^2 f(\boldsymbol{x}_0) + \sum_{j=1}^{m} \alpha_j \nabla^2 h_j(\boldsymbol{x}_0) + \sum_{l=1}^{p} \beta_l \nabla^2 g_l(\boldsymbol{x}_0)\right) \boldsymbol{y} \geq \gamma$ *for all* $\boldsymbol{y} \in \mathcal{R}^n$ *satisfying* $|\boldsymbol{y}|_0 = 1$, $\left|\nabla h_j(\boldsymbol{x}_0)\boldsymbol{y}\right|_0 < \epsilon$ *for all* $j \in \{1, \ldots, m\}$, $|\nabla g_l(\boldsymbol{x}_0)\boldsymbol{y}|_0 < \epsilon$ *for all* $l \in L = \{k : \beta_k > 0\}$ *and* $\nabla g_l(\boldsymbol{x}_0)\boldsymbol{y} < \epsilon$ *for all* $l \in I(\boldsymbol{x}_0) \setminus L$, *where* $I(\boldsymbol{x}_0) = \{k : g_k(\boldsymbol{x}_0) = 0\}$.

Then $\boldsymbol{x}_0$ is a strict local minimum for f under the constraints of (8.1).

9 Computational Applications

The general question of efficient differentiation is at the core of many parts of the work on perturbation and aberration theories relevant in Physics and Engineering. In this case, derivatives of highly complicated functions have to be computed to high orders. However, even when the derivative of the function is known to exist at the given point, numerical methods fail to give an accurate value of the derivative; the error increases with the order, and for orders greater than three, the errors often become too large for the results to be practically useful.

On the other hand, while formula manipulators like Mathematica are successful in finding low-order derivatives of simple functions, they fail for high-order derivatives of very complicated functions. Moreover, they fail to find the derivatives of certain functions at given points even though the functions are differentiable at the respective points. This is generally connected to the occurrence of non-differentiable parts that do not affect the differentiability of the end result as well as the occurrence of branch points in coding as in IF-ELSE structures.

Using Calculus on $\mathcal{R}$ and the fact that the field has infinitely small numbers represents a new method for computational differentiation that avoids the well-known accuracy problems of numerical differentiation tools. It also avoids the often rather stringent limitations of formula manipulators that restrict the complexity of the function that can be differentiated, and the orders to which differentiation can be performed.

By a computer function, we denote any real-valued function that can be typed on a computer. The $\mathcal{R}$ numbers as well as the continuations to $\mathcal{R}$ of the intrinsic functions (and hence of all computer functions) have all been implemented for use on a computer, using the code COSY INFINITY [8]. Using the calculus on $\mathcal{R}$, we formulate a necessary and sufficient condition for the derivatives of a computer function to exist, and show how to find these derivatives whenever they exist [43, 45]. The new technique of computing the derivatives of computer functions, which we summarize below, achieves results that combine the accuracy of formula manipulators with the speed of classical numerical methods, that is the best of both worlds. The method is much faster than Mathematica and other formula manipulators since no symbolic differentiation is required before the numerical evaluation of the derivatives. Moreover, the results obtained are accurate up to machine precision—the error is infinitely small and hence it does not mix with the real derivative; this represents a clear advantage over traditional numerical differentiation methods in which case finite errors result from digit cancelation in the floating point representation and for high orders the errors usually become too large for the results to be of any practical use.

Lemma 9.1 *Let f be a computer function. Then f is defined at x_0 if and only if $f(x_0)$ can be computed on a computer.*

This lemma hinges on a careful implementation of the intrinsic functions and operations, in particular in the sense that they should be executable for any floating point number in the domain of definition that produces a result within the range of allowed floating point numbers.

Lemma 9.2 *Let f be a computer function, and let x_0 be a real number. Then f is right-continuous at x_0 if and only if $f(x_0)$ and $f(x_0 + d)$ are defined, and $f(x_0 + d) =_0 f(x_0)$.*

f is left-continuous at x_0 if and only if $f(x_0)$ and $f(x_0 - d)$ are defined, and $f(x_0 - d) =_0 f(x_0)$.

Finally, f is continuous at x_0 if and only if it is both right-continuous and left-continuous at x_0; that is, if and only if $f(x_0 - d)$, $f(x_0)$, and $f(x_0 + d)$ are all defined, and

$$f(x_0 - d) =_0 f(x_0) =_0 f(x_0 + d).$$

Theorem 9.3 *Let f be a computer function that is continuous at x_0. Then f is differentiable at x_0 if and only if*

$$\frac{f(x_0 + d) - f(x_0)}{d} \text{ and } \frac{f(x_0) - f(x_0 - d)}{d}$$

are both at most finite in (ordinary) absolute value, and their real parts agree. In this case,

$$\frac{f(x_0 + d) - f(x_0)}{d} =_0 f'(x_0) =_0 \frac{f(x_0) - f(x_0 - d)}{d}.$$

If f is differentiable at x_0, then f is twice differentiable at x_0 if and only if

$$\frac{f(x_0 + 2d) - 2f(x_0 + d) + f(x_0)}{d^2} \text{ and } \frac{f(x_0) - 2f(x_0 - d) + f(x_0 - 2d)}{d^2}$$

are both at most finite in (ordinary) absolute value, and their real parts agree. In this case

$$\frac{f(x_0 + 2d) - 2f(x_0 + d) + f(x_0)}{d^2} =_0 f''(x_0) =_0 \frac{f(x_0) - 2f(x_0 - d) + f(x_0 - 2d)}{d^2}.$$

In general, if f is $(n - 1)$ times differentiable at x_0, then f is n times differentiable at x_0 if and only if

$$\frac{\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j} f(x_0 + jd)}{d^n} \text{ and } \frac{\sum_{j=0}^{n}(-1)^{j}\binom{n}{j} f(x_0 - jd)}{d^n}$$

are both at most finite in (ordinary) absolute value, and their real parts agree. In this case,

$$\frac{\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j} f(x_0 + jd)}{d^n} =_0 f^{(n)}(x_0) =_0 \frac{\sum_{j=0}^{n}(-1)^{j}\binom{n}{j} f(x_0 - jd)}{d^n}.$$

Since knowledge of $f(x_0 - d)$ and $f(x_0 + d)$ gives us all the information about a computer function f in an interval $(x_0 - \sigma, x_0 + \sigma)$, with real $\sigma > 0$, around x_0, we have the following result which states that, from the mere knowledge of $f(x_0 - d)$ and $f(x_0 + d)$, we can find at once the order of differentiability of f at x_0 and the accurate values of all existing derivatives.

Theorem 9.4 *Let f be a computer function that is defined at x_0; and let $n \in \mathbb{N}$ be given. Then f is n times differentiable at x_0 if and only if $f(x_0 - d)$ and $f(x_0 + d)$ are both defined and can be written as*

$$f(x_0 - d) =_n f(x_0) + \sum_{j=1}^{n} (-1)^j \alpha_j d^j \text{ and } f(x_0 + d) =_n f(x_0) + \sum_{j=1}^{n} \alpha_j d^j,$$

where the α_j's are real numbers. Moreover, in this case $f^{(j)}(x_0) = j!\,\alpha_j$ for $1 \le j \le n$.

Now consider, as an example, the function

$$g(x) = \frac{\sin\left(x^3 + 2x + 1\right) + \frac{3+\cos(\sin(\ln|1+x|))}{\exp\left(\tanh\left(\sinh\left(\cosh\left(\frac{\sin(\cos(\tan(\exp(x))))}{\cos(\sin(\exp(\tan(x+2))))}\right)\right)\right)\right)}}{2 + \sin\left(\sinh\left(\cos\left(\tan^{-1}\left(\ln\left(\exp(x) + x^2 + 3\right)\right)\right)\right)\right)}. \tag{9.1}$$

Using the $\mathcal{R}$ calculus, we find $g^{(n)}(0)$ for $0 \le n \le 10$. These numbers are listed in Table 1; we note that, for $0 \le n \le 10$, we list the CPU time needed to obtain all derivatives of g at 0 up to order n and not just $g^{(n)}(0)$. For comparison purposes, we give in Table 2 the function value and the first six derivatives computed with Mathematica. Note that the respective values listed in Tables 1 and 2 agree. However, Mathematica used much more CPU time to compute the first six derivatives, and it failed to find the seventh derivative as it ran out of memory. We also list in Table 3 the first ten derivatives of g at 0 computed numerically using the numerical differentiation formulas

$$g^{(n)}(0) = (\Delta x)^{-n} \left(\sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} g(j\Delta x) \right), \quad \Delta x = 10^{-16/(n+1)},$$

Table 1 $g^{(n)}(0), 0 \le n \le 10$, computed with $\mathcal{R}$ calculus

Order n	$g^{(n)}(0)$	CPU Time
0	1.004845319007115	1.820 ms
1	0.4601438089634254	2.070 ms
2	−5.266097568233224	3.180 ms
3	−52.82163351991485	4.830 ms
4	−108.4682847837855	7.700 ms
5	16451.44286410806	11.640 ms
6	541334.9970224757	18.050 ms
7	7948641.189364974	26.590 ms
8	−144969388.2104904	37.860 ms
9	−15395959663.01733	52.470 ms
10	−618406836695.3634	72.330 ms

Table 2 $g^{(n)}(0), 0 \leq n \leq 6$, computed with Mathematica

Order n	$g^{(n)}(0)$	CPU Time
0	1.004845319007116	0.11 s
1	0.4601438089634254	0.17 s
2	−5.266097568233221	0.47 s
3	−52.82163351991483	2.57 s
4	−108.4682847837854	14.74 s
5	16451.44286410805	77.50 s
6	541334.9970224752	693.65 s

Table 3 $g^{(n)}(0), 1 \leq n \leq 10$, computed numerically

Order n	$g^{(n)}(0)$	Relative Error
1	0.4601437841866840	54×10^{-9}
2	−5.266346392944456	47×10^{-6}
3	−52.83767867680922	30×10^{-5}
4	−87.27214664649106	0.20
5	19478.29555909866	0.18
6	633008.9156614641	0.17
7	−12378052.73279768	2.6
8	−1282816703.632099	7.8
9	83617811421.48561	6.4
10	91619495958355.24	149

for $1 \leq n \leq 10$, together with the corresponding relative errors obtained by comparing the numerical values with the respective exact values computed using $\mathcal{R}$ calculus.

On the other hand, formula manipulators fail to find the derivatives of certain functions at given points even though the functions are differentiable at the respective points. For example, the functions

$$g_1(x) = |x|^{5/2} \cdot g(x) \text{ and } g_2(x) = \begin{cases} \frac{1-\exp(-x^2)}{x} \cdot g(x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases},$$

where $g(x)$ is the function given in Eq. (9.1), are both differentiable at 0; but the attempt to compute their derivatives using formula manipulators fails. This is not specific to g_1 and g_2, and is generally connected to the occurrence of non-differentiable parts that do not affect the differentiability of the end result, of which case g_1 is an example, as well as the occurrence of branch points in coding as in IF-ELSE structures, of which case g_2 is an example.

More recently, in [16, 18], building on the success of computational differentiation above, we succeeded in achieving more computational applications of the Levi-Civita numbers such as numerical integration as well as the computation of numerical sequences that are given by generating functions like the Bernoulli numbers.

In the following section we give a brief summary of our work on developing a non-Archimedean operator theory on a Banach space over the complex Levi-Civita field $\mathcal{C}$ which is the result of a recent collaboration with José Aguayo (Universidad de Concepción, Chile) and Miguel Nova (Universidad Católica de la Santísima, Concepción, Chile). For lack of space, we will omit all the details here and refer the interested reader to [1–3].

10 Non-Archimedean Operator Theory

Let c_0 denote the space of all null sequences of elements in $\mathcal{C}$. The natural inner product on c_0 induces the sup-norm of c_0. In [1], we show that c_0 is not orthomodular then we characterize those closed subspaces of c_0 with an orthonormal complement with respect to the inner product. Such a subspace, together with its orthonormal complement, defines a special kind of projection, the normal projection. We present characterizations of normal projections as well as other kinds of operators, the self-adjoint and compact operators on c_0. In [2], we work on some B*-algebras of operators, including those mentioned above; and we define an inner product on such algebras that induces the usual norm of operators. Finally, in [3], we study the properties of positive operators on c_0 which are similar to those of positive operators in classical functional analysis; however the proofs of many of the results are nonclassical. Then we use our study of positive operators to introduce a partial order on the set of compact and self-adjoint operators on c_0 and study the properties of that partial order.

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