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# The implicit function theorem in a non-Archimedean setting ☆

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ABSTRACT

In this paper, the inverse function theorem and the implicit function theorem in a non-Archimedean setting will be discussed. We denote by $\mathcal{N}$ any non-Archimedean field extension of the real numbers that is real closed and Cauchy complete in the topology induced by the order; and we study the properties of locally uniformly differentiable functions from $\mathcal{N}^n$ to $\mathcal{N}^m$. Then we use that concept of local uniform differentiability to formulate and prove the inverse function theorem for functions from $\mathcal{N}^n$ to $\mathcal{N}^n$ and the implicit function theorem for functions from $\mathcal{N}^n$ to $\mathcal{N}^m$ with $m < n$.

## 1. INTRODUCTION

We start this section by reviewing some basic terminology and facts about non-Archimedean fields. So let $F$ be an ordered non-Archimedean field extension of $\mathbb{R}$. We introduce the following terminology.

**Definition 1.1** ($\sim$, $\approx$, $\ll$, $H$, $\lambda$). For $x, y \in F^* := F \setminus \{0\}$, we say $x \sim y$ if there exist $n, m \in \mathbb{N}$ such that $n|x| > |y|$ and $m|y| > |x|$; for nonnegative $x, y \in F$, we say that $x$ is infinitely smaller than $y$ and write $x \ll y$ if $nx < y$ for all $n \in \mathbb{N}$, and we

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say that $x$ is infinitely small if $x \ll 1$ and $x$ is finite if $x \sim 1$; finally, we say that $x$ is approximately equal to $y$ and write $x \approx y$ if $x \sim y$ and $|x - y| \ll |x|$. We also set $\lambda(x) = [x]$, the class of $x$ under the equivalence relation $\sim$.

The set of equivalence classes $H$ (under the relation $\sim$) is naturally endowed with an addition via $[x] + [y] = [x \cdot y]$ and an order via $[x] < [y]$ if $|y| \ll |x|$ (or $|x| \gg |y|$), both of which are readily checked to be well-defined. It follows that $(H, +, <)$ is an ordered group, often referred to as the Hahn group or skeleton group, whose neutral element is [1], the class of 1. It follows from the above that the projection $\lambda$ from $F^*$ to $H$ is a valuation.

The theorem of Hahn [2] provides a complete classification of non-Archimedean extensions of $\mathbb{R}$ in terms of their skeleton groups. In fact, invoking the axiom of choice it is shown that the elements of any such field $F$ can be written as formal power series over its skeleton group $H$ with real coefficients, and the set of appearing exponents forms a well-ordered subset of $H$.

From general properties of formal power series fields [6,8], it follows that if $H$ is divisible then $F$ is real closed; that is, every positive element of $F$ is a square in $F$ and every polynomial of odd degree over $F$ has at least one root in $F$. For a general overview of the algebraic properties of formal power series fields, we refer to the comprehensive overview by Ribenboim [9], and for an overview of the related valuation theory the book by Krull [3]. A thorough and complete treatment of ordered structures can also be found in [7].

Throughout this paper, we will denote by $\mathcal{N}$ any totally ordered non-Archimedean field extension of $\mathbb{R}$ that is complete in the order topology and whose skeleton group $H$ is Archimedean, i.e. a subgroup of $\mathbb{R}$. The smallest such field is the field $L$ of the formal Laurent series whose skeleton group is $H = \mathbb{Z}$; and the smallest such field that is also real closed is the Levi-Civita field $\mathcal{R}$, first introduced in [4, 5]. In this case $H = \mathbb{Q}$, and for any element $x \in \mathcal{R}$, the set of exponents in the Hahn representation of $x$ is a left-finite subset of $\mathbb{Q}$, i.e. below any rational bound $r$ there are only finitely many exponents. The Levi-Civita field $\mathcal{R}$ is of particular interest because of its practical usefulness. Since the supports of the elements of $\mathcal{R}$ (when viewed as maps from $H = \mathbb{Q}$ into $\mathbb{R}$) are left-finite, it is possible to represent these numbers on a computer [1]. Having infinitely small numbers, the errors in classical numerical methods can be made infinitely small and hence irrelevant in all practical applications. One such application is the computation of derivatives of real functions representable on a computer [12], where both the accuracy of formula manipulators and the speed of classical numerical methods are achieved. For a review of the Levi-Civita field $\mathcal{R}$, see [1,10–21].

In this paper, we will generalize the inverse function theorem and the implicit function theorem from real calculus to, respectively, functions from an open set $A \subset \mathcal{N}^n$ to $\mathcal{N}^n$ and functions from $A \subset \mathcal{N}^n$ to $\mathcal{N}^m$ $(1 \leqslant m < n)$. Because of the total disconnectedness of $\mathcal{N}$ in the order topology, a stronger condition on the function than that of the real case is needed for the proof of both theorems. We introduce the concept of local uniform differentiability and we show that a locally uniformly differentiable function from $A \subset \mathcal{N}^n$ to $\mathcal{N}^m$ is $C^1$ on $A$. Then we show that if we require the function to be locally uniformly differentiable rather than $C^1$ as in real calculus, then we can state and prove similar versions of the inverse function theorem and the implicit function theorem in this non-Archimedean context.

## 2. LINEAR TRANSFORMATIONS

In this section we review the properties of linear transformations from $\mathcal{N}^n$ into $\mathcal{N}^m$, which are similar to those of linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$.

**Proposition 2.1.** *Let* $\mathbf{L}:\mathcal{N}^n \to \mathcal{N}^m$ *be a linear transformation. Then* $\{|\mathbf{L}(\mathbf{t})|\colon |\mathbf{t}| \leqslant 1\}$ *is bounded.*

**Proof.** Let

$$\underline{\mathbf{L}} = \begin{pmatrix} L_{11} & L_{12} & \dots & L_{1n} \\ L_{21} & L_{22} & \dots & L_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ L_{m1} & L_{m2} & \dots & L_{mn} \end{pmatrix}$$

denote the matrix of the linear transformation $\mathbf{L}$, and let $\alpha = \max\{|L_{ij}|\colon i = 1, \dots, m;\ j = 1, \dots, n\}$. Then for $|\mathbf{t}| \leqslant 1$ we have that

$$\begin{aligned} |\mathbf{L}(\mathbf{t})| = |\underline{\mathbf{L}}\mathbf{t}| &= \left| \begin{pmatrix} L_{11} & L_{12} & \dots & L_{1n} \\ L_{21} & L_{22} & \dots & L_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ L_{m1} & L_{m2} & \dots & L_{mn} \end{pmatrix} \begin{pmatrix} t_1 \\ t_2 \\ \vdots \\ t_n \end{pmatrix} \right| = \left| \begin{pmatrix} \sum_{j=1}^n L_{1j} t_j \\ \vdots \\ \vdots \\ \sum_{j=1}^n L_{mj} t_j \end{pmatrix} \right| \\ &= \sqrt{\sum_{i=1}^m \left( \sum_{j=1}^n L_{ij} t_j \right)^2} \leqslant \sqrt{\sum_{i=1}^m \left( \sum_{j=1}^n |L_{ij}||t_j| \right)^2} \\ &\leqslant \sqrt{\sum_{i=1}^m \left( \sum_{j=1}^n \alpha \cdot 1 \right)^2} = \sqrt{\sum_{i=1}^m (n\alpha)^2} = \sqrt{m} n\alpha. \end{aligned}$$

Thus, $\{|\mathbf{L}(\mathbf{t})|\colon |\mathbf{t}| \leqslant 1\}$ is bounded above by $\sqrt{m} n\alpha$. □

**Corollary 2.2.** *Let* $\mathbf{L}:\mathcal{N}^n \to \mathcal{N}^m$ *be a linear transformation and let* $\mathfrak{L}$ *be an upper bound for* $\{|\mathbf{L}(\mathbf{t})|\colon |\mathbf{t}| \leqslant 1\}$. *Then* $|\mathbf{L}(\mathbf{t})| \leqslant \mathfrak{L}|\mathbf{t}|\ \forall \mathbf{t} \in \mathcal{N}^n$.

**Proof.** Let $\mathbf{t} \in \mathcal{N}^n$. If $\mathbf{t} = \mathbf{0}$, then $|\mathbf{L}(\mathbf{t})| = 0 = \mathfrak{L}|\mathbf{t}|$ and we are done. Otherwise, let $c = |\mathbf{t}|^{-1}$; then $|c\mathbf{t}| = 1$, and so $c|\mathbf{L}(\mathbf{t})| = |c\mathbf{L}(\mathbf{t})| = |\mathbf{L}(c\mathbf{t})| \leqslant \mathfrak{L}$. Thus, $|\mathbf{L}(\mathbf{t})| \leqslant \mathfrak{L}|\mathbf{t}|$. □

**Corollary 2.3.** *Let* $\mathbf{L}:\mathcal{N}^n \to \mathcal{N}^n$ *be an invertible linear transformation and let* $\bar{\mathfrak{L}}$ *be an upper bound for* $\{|\mathbf{L}^{-1}(\mathbf{t})|\colon |\mathbf{t}| \leqslant 1\}$. *Then* $|\mathbf{L}(\mathbf{t})| \geqslant \frac{|\mathbf{t}|}{\bar{\mathfrak{L}}}\ \forall \mathbf{t} \in \mathcal{N}^n$.

**Proof.** First we note that, since $\mathbf{L}^{-1}$ is invertible, $\mathbf{L}^{-1}(\mathbf{t}) = \mathbf{0}$ only if $\mathbf{t} = \mathbf{0}$; and hence $\bar{\mathfrak{L}} > 0$. Now let $\mathbf{t} \in \mathcal{N}^n$ be given. Then $|\mathbf{t}| = |\mathbf{L}^{-1}(\mathbf{L}(\mathbf{t}))| \leqslant \bar{\mathfrak{L}}|\mathbf{L}(\mathbf{t})|$; and hence $|\mathbf{L}(\mathbf{t})| \geqslant \frac{|\mathbf{t}|}{\bar{\mathfrak{L}}}$. □

**Lemma 2.4.** *Let* $\mathbf{g}:\mathcal{N}^n \to \mathcal{N}^m$ *be* $C^1$*; and let* $\mathfrak{L}$ *be an upper bound for* $\{|\mathbf{Dg}(\mathbf{x}_0)(\mathbf{x})|\colon\ |\mathbf{x}| \leqslant 1\}$*, where* $\mathbf{Dg}(\mathbf{x}_0)$ *denotes the linear map from* $\mathcal{N}^n$ *to* $\mathcal{N}^m$ *defined by the* $m \times n$ *Jacobian matrix of* $\mathbf{g}$ *at* $\mathbf{x}_0$*:*

$$\begin{pmatrix} \mathbf{g}_1^1(\mathbf{x}_0) & \mathbf{g}_2^1(\mathbf{x}_0) & \dots & \mathbf{g}_n^1(\mathbf{x}_0) \\ \mathbf{g}_1^2(\mathbf{x}_0) & \mathbf{g}_2^2(\mathbf{x}_0) & \dots & \mathbf{g}_n^2(\mathbf{x}_0) \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{g}_1^m(\mathbf{x}_0) & \mathbf{g}_2^m(\mathbf{x}_0) & \dots & \mathbf{g}_n^m(\mathbf{x}_0) \end{pmatrix}$$

*with* $\mathbf{g}_j^i(\mathbf{x}_0) = \frac{\partial g_i}{\partial x_j}(\mathbf{x}_0)$ *for* $1 \leqslant i \leqslant m$ *and* $1 \leqslant j \leqslant n$*. Then* $\forall \epsilon > 0$ *in* $\mathcal{N}$*,* $\exists \delta > 0$ *in* $\mathcal{N}$ *such that* $|\mathbf{Dg}(\mathbf{y})(\mathbf{x})| < (\mathfrak{L} + \epsilon)|\mathbf{x}|$ $\forall \mathbf{y} \in B_\delta(\mathbf{x}_0)$ *and* $\forall \mathbf{x} \in \mathcal{N}^n$*.*

**Proof.** Let $\epsilon > 0$ in $\mathcal{N}$ be given. Since $\mathbf{g}$ is $C^1$ then for every $i$ and $j$, $\exists \delta_j^i > 0$ such that $|\mathbf{g}_j^i(\mathbf{x}_0) - \mathbf{g}_j^i(\mathbf{y})| < \frac{\epsilon}{n\sqrt{m}}$ whenever $\mathbf{y} \in B_{\delta_j^i}(\mathbf{x}_0)$. Let $\delta = \min\{\delta_j^i\colon\ i = 1, \dots, m;\ j = 1, \dots, n\}$. Then, using the proof of Proposition 2.1, for all $\mathbf{y} \in B_\delta(\mathbf{x}_0)$ we have that $\{|\mathbf{Dg}(\mathbf{x}_0)(\mathbf{x}) - \mathbf{Dg}(\mathbf{y})(\mathbf{x})|\colon\ |\mathbf{x}| \leqslant 1\}$ is bounded above by $\epsilon$. Thus, by Corollary 2.2, we have that $|\mathbf{Dg}(\mathbf{x}_0)(\mathbf{x}) - \mathbf{Dg}(\mathbf{y})(\mathbf{x})| < \epsilon|\mathbf{x}|$ $\forall \mathbf{y} \in B_\delta(\mathbf{x}_0)$ and $\forall \mathbf{x} \in \mathcal{N}^n$. Therefore, $|\mathbf{Dg}(\mathbf{y})(\mathbf{x})| < \epsilon|\mathbf{x}| + |\mathbf{Dg}(\mathbf{x}_0)(\mathbf{x})|$; and hence,

$$|\mathbf{Dg}(\mathbf{y})(\mathbf{x})| < (\epsilon + \mathfrak{L})|\mathbf{x}|.$$ □

In the following section, we introduce the concept of local uniform differentiability, and we review the properties of locally uniformly differentiable functions from an open subset $A$ of $\mathcal{N}^n$ into $\mathcal{N}^m$.

## 3. LOCAL UNIFORM DIFFERENTIABILITY

In the rest of the paper, let $A$ denote an open subset of $\mathcal{N}^n$; consequently, whenever we speak of a ball $B_\delta(\mathbf{x})$ around a point $\mathbf{x}$ in $A$ it is assumed that $\delta > 0$ is small enough so that $B_\delta(\mathbf{x}) \subset A$.

**Definition 3.1** (Uniformly differentiable). Let $\mathbf{f}\colon A \to \mathcal{N}^m$ be differentiable on $A$. Then we say that $\mathbf{f}$ is uniformly differentiable on $A$ if $\forall \epsilon > 0$ in $\mathcal{N}$, $\exists \delta > 0$ in $\mathcal{N}$ such that whenever $\mathbf{x}, \mathbf{y} \in A$ and $|\mathbf{y} - \mathbf{x}| < \delta$ we have that $|\mathbf{f}(\mathbf{y}) - \mathbf{f}(\mathbf{x}) - \mathbf{Df}(\mathbf{x})(\mathbf{y} - \mathbf{x})| < \epsilon|\mathbf{y} - \mathbf{x}|$.

**Definition 3.2** (Locally uniformly differentiable). Let $\mathbf{f}\colon A \to \mathcal{N}^m$ be differentiable on $A$. Then we say that $\mathbf{f}$ is locally uniformly differentiable on $A$ if $\forall \mathbf{x} \in A$, $\exists \delta_{\mathbf{x}} > 0$ in $\mathcal{N}$ such that $\mathbf{f}$ is uniformly differentiable on $B_{\delta_{\mathbf{x}}}(\mathbf{x})$.

**Proposition 3.3.** *Let* $\mathbf{f}: A \to \mathcal{N}^m$ *be differentiable on* $A$*. Then* $\mathbf{f}$ *is continuous on* $A$*.*

**Proof.** Let $\mathbf{x} \in A$ and let $\mathfrak{L}_{\mathbf{x}} > 0$ be an upper bound for $\{|\mathbf{Df}(\mathbf{x})(\mathbf{y})|: |\mathbf{y}| \leqslant 1\}$. Since $\mathbf{f}$ is differentiable at $\mathbf{x}$, $\exists \delta_0 > 0$ in $\mathcal{N}$ such that whenever $\mathbf{y} \in B_{\delta_0}(\mathbf{x})$, we have that $|\mathbf{f}(\mathbf{y}) - \mathbf{f}(\mathbf{x}) - \mathbf{Df}(\mathbf{x})(\mathbf{y}-\mathbf{x})| \leqslant \mathfrak{L}_{\mathbf{x}}|\mathbf{y}-\mathbf{x}|$. Let $\epsilon > 0$ in $\mathcal{N}$ be given. Let $\delta = \min\{\delta_0, \frac{\epsilon}{2\mathfrak{L}_{\mathbf{x}}}\}$. Then for $\mathbf{y} \in B_\delta(\mathbf{x})$ we have that

$$\begin{aligned}
|\mathbf{f}(\mathbf{y}) - \mathbf{f}(\mathbf{x})| &= |\mathbf{f}(\mathbf{y}) - \mathbf{f}(\mathbf{x}) - \mathbf{Df}(\mathbf{x})(\mathbf{y}-\mathbf{x}) + \mathbf{Df}(\mathbf{x})(\mathbf{y}-\mathbf{x})| \\
&\leqslant |\mathbf{f}(\mathbf{y}) - \mathbf{f}(\mathbf{x}) - \mathbf{Df}(\mathbf{x})(\mathbf{y}-\mathbf{x})| + |\mathbf{Df}(\mathbf{x})(\mathbf{y}-\mathbf{x})| \\
&\leqslant \mathfrak{L}_{\mathbf{x}}|\mathbf{y}-\mathbf{x}| + \mathfrak{L}_{\mathbf{x}}|\mathbf{y}-\mathbf{x}| \\
&= 2\mathfrak{L}_{\mathbf{x}}|\mathbf{y}-\mathbf{x}| \\
&< 2\mathfrak{L}_{\mathbf{x}}\delta \\
&\leqslant \epsilon.
\end{aligned}$$

□

**Theorem 3.4.** *Let* $\mathbf{f}: A \to \mathcal{N}^m$ *be locally uniformly differentiable on* $A$*. Then* $\mathbf{f}$ *is* $C^1$ *on* $A$*.*

**Proof.** Let $\mathbf{x}_0 \in A$. By Proposition 3.3, $\mathbf{f}$ is continuous at $\mathbf{x}_0$. Let $\delta_{x_0} > 0$ in $\mathcal{N}$ be such that $\mathbf{f}$ is uniformly differentiable on $B_{\delta_{\mathbf{x}_0}}(\mathbf{x}_0)$. Now let $\epsilon > 0$ in $\mathcal{N}$ be given. Then $\exists \delta_1 > 0$ in $\mathcal{N}$ such that for $\mathbf{s}, \mathbf{t} \in B_{\delta_{\mathbf{x}_0}}(\mathbf{x}_0)$ we have that $|\mathbf{f}(\mathbf{s}) - \mathbf{f}(\mathbf{t}) - \mathbf{Df}(\mathbf{t})(\mathbf{s} - \mathbf{t})| \leqslant \frac{\epsilon}{4}|\mathbf{s}-\mathbf{t}|$ whenever $|\mathbf{s}-\mathbf{t}| \leqslant \delta_1$. Let $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$ be given. Then it follows that

$$\left|f^i(\mathbf{s}) - f^i(\mathbf{t}) - \mathbf{D}f^i(\mathbf{t})(\mathbf{s}-\mathbf{t})\right| \leqslant \frac{\epsilon}{4}|\mathbf{s}-\mathbf{t}|.$$

Let $\delta_2 = \min\{\delta_1, \frac{\delta_{x_0}}{2}\}$. Then for any $\mathbf{x} \in B_{\delta_2}(\mathbf{x}_0)$ we have that

$$\left|f^i(\mathbf{x} + \delta_2\hat{\mathbf{e}}_j) - f^i(\mathbf{x}) - \mathbf{D}f^i(\mathbf{x})(\delta_2\hat{\mathbf{e}}_j)\right| \leqslant \frac{\epsilon\delta_2}{4}.$$

In other words,

$$\left|f^i(\mathbf{x} + \delta_2\hat{\mathbf{e}}_j) - f^i(\mathbf{x}) - f^i_j(\mathbf{x})\delta_2\right| \leqslant \frac{\epsilon\delta_2}{4}.$$

Now, since $f^i$ is continuous on $B_{\delta_{\mathbf{x}_0}}(\mathbf{x}_0)$, $\exists \delta_3 > 0$ such that $\forall \mathbf{s} \in B_{\delta_3}(\mathbf{x}_0)$, we have that

$$\left|f^i(\mathbf{s}) - f^i(\mathbf{x}_0)\right| \leqslant \frac{\epsilon\delta_2}{4}.$$

Additionally, $\exists \delta_4 > 0$ such that $\forall \mathbf{s} \in B_{\delta_4}(\mathbf{x}_0 + \delta_2\hat{\mathbf{e}}_j)$, we have that

$$\left|f^i(\mathbf{s}) - f^i(\mathbf{x}_0 + \delta_2\hat{\mathbf{e}}_j)\right| \leqslant \frac{\epsilon\delta_2}{4}.$$

Let $\delta = \min\{\delta_2, \delta_3, \delta_4\}$ and let $\mathbf{y} \in B_\delta(\mathbf{x}_0)$. Then we have that

$$
\begin{aligned}
&\left|f_j^i(\mathbf{y})\delta_2 - f_j^i(\mathbf{x}_0)\delta_2\right| \\
&\quad = \big|f^i(\mathbf{x}_0) - f^i(\mathbf{x}_0 + \delta_2\hat{\mathbf{e}}_j) - f_j^i(\mathbf{x}_0)\delta_2 - f^i(\mathbf{y}) \\
&\qquad + f^i(\mathbf{y} + \delta_2\hat{\mathbf{e}}_j) + f_j^i(\mathbf{y})\delta_2 - f^i(\mathbf{x}_0) \\
&\qquad + f^i(\mathbf{x}_0 + \delta_2\hat{\mathbf{e}}_j) + f^i(\mathbf{y}) - f^i(\mathbf{y} + \delta_2\hat{\mathbf{e}}_j)\big| \\
&\quad \leqslant \left|f^i(\mathbf{x}_0 + \delta_2\hat{\mathbf{e}}_j) - f^i(\mathbf{x}_0) - f_j^i(\mathbf{x}_0)\delta_2\right| \\
&\qquad + \left|f^i(\mathbf{y} + \delta_2\hat{\mathbf{e}}_j) - f^i(\mathbf{y}) - f_j^i(\mathbf{y})\delta_2\right| \\
&\qquad + \left|f^i(\mathbf{x}_0) - f^i(\mathbf{y})\right| + \left|f^i(\mathbf{x}_0 + \delta_2\hat{\mathbf{e}}_j) - f^i(\mathbf{y} + \delta_2\hat{\mathbf{e}}_j)\right| \\
&\quad \leqslant \frac{\epsilon\delta_2}{4} + \frac{\epsilon\delta_2}{4} + \frac{\epsilon\delta_2}{4} + \frac{\epsilon\delta_2}{4} \\
&\quad = \epsilon\delta_2.
\end{aligned}
$$

Thus, for $\mathbf{y} \in B_\delta(\mathbf{x}_0)$ we have that

$$\left|f_j^i(\mathbf{y}) - f_j^i(\mathbf{x}_0)\right| \leqslant \epsilon;$$

and hence $\mathbf{f}$ is $C^1$ on $A$. □

**Remark 3.5.** Theorem 3.4 shows that the class of locally uniformly differentiable functions is a subset of the class of $C^1$ functions. However, this is still large enough to include all polynomial functions as Corollary 3.13 and Corollary 3.14 below will show.

**Lemma 3.6.** *Let* $\mathbf{f} : A \to \mathcal{N}^m$ *be locally uniformly differentiable on* $A$*. Then* $\forall \mathbf{x} \in A$ *we have that*

$$\text{(3.1)} \qquad \forall \epsilon > 0\ \exists \delta > 0 \ni \mathbf{s}, \mathbf{t} \in B_\delta(\mathbf{x}) \Longrightarrow |\mathbf{f}(\mathbf{s}) - \mathbf{f}(\mathbf{t}) - D\mathbf{f}(\mathbf{x})(\mathbf{s} - \mathbf{t})| < \epsilon|\mathbf{s} - \mathbf{t}|.$$

**Proof.** Suppose that $\mathbf{f}$ is locally uniformly differentiable on $A$, and let $\mathbf{x} \in A$. Let $\epsilon > 0$ in $\mathcal{N}$ be given. Let $\delta_{\mathbf{x}} > 0$ in $\mathcal{N}$ be such that $\mathbf{f}$ is uniformly differentiable on $B_{\delta_{\mathbf{x}}}(\mathbf{x})$. Since $\mathbf{f}$ is uniformly differentiable on $B_{\delta_{\mathbf{x}}}(\mathbf{x})$, $\exists \delta_1 > 0$ such that $\forall \mathbf{s}, \mathbf{t} \in B_{\delta_{\mathbf{x}}}(\mathbf{x})$ satisfying $|\mathbf{s} - \mathbf{t}| < \delta_1$, we have that $|\mathbf{f}(\mathbf{t}) - \mathbf{f}(\mathbf{s}) - \mathbf{Df}(\mathbf{s})(\mathbf{t} - \mathbf{s})| < \frac{\epsilon}{2}|\mathbf{t} - \mathbf{s}|$. Additionally, by Theorem 3.4, $\mathbf{f}$ is $C^1$ on $A$, so $\exists \delta_2 > 0$ such that $\forall \mathbf{s} \in B_{\delta_2}(\mathbf{x})$, we have that $|\mathbf{Df}(\mathbf{x})(\mathbf{t} - \mathbf{s}) - \mathbf{Df}(\mathbf{s})(\mathbf{t} - \mathbf{s})| < \frac{\epsilon}{2}|\mathbf{t} - \mathbf{s}|$.

Let $\delta = \min\{\frac{\delta_1}{2}, \delta_2, \delta_{\mathbf{x}}\}$. Then for $\mathbf{s}, \mathbf{t} \in B_\delta(\mathbf{x})$ we have that

$$
\begin{aligned}
|\mathbf{f}(\mathbf{t}) - \mathbf{f}(\mathbf{s}) - \mathbf{Df}(\mathbf{x})(\mathbf{t} - \mathbf{s})| &= |\mathbf{f}(\mathbf{t}) - \mathbf{f}(\mathbf{s}) - \mathbf{Df}(\mathbf{s})(\mathbf{t} - \mathbf{s}) \\
&\quad + \mathbf{Df}(\mathbf{s})(\mathbf{t} - \mathbf{s}) - \mathbf{Df}(\mathbf{x})(\mathbf{t} - \mathbf{s})| \\
&\leqslant |\mathbf{f}(\mathbf{t}) - \mathbf{f}(\mathbf{s}) - \mathbf{Df}(\mathbf{s})(\mathbf{t} - \mathbf{s})| \\
&\quad + |\mathbf{Df}(\mathbf{x})(\mathbf{t} - \mathbf{s}) - \mathbf{Df}(\mathbf{s})(\mathbf{t} - \mathbf{s})| \\
&\leqslant \frac{\epsilon}{2}|\mathbf{s} - \mathbf{t}| + \frac{\epsilon}{2}|\mathbf{s} - \mathbf{t}| \\
&= \epsilon|\mathbf{s} - \mathbf{t}|.
\end{aligned}
$$

Thus $\mathbf{f}$ satisfies (3.1). $\square$

**Proposition 3.7.** *Let* $\mathbf{L}:\mathcal{N}^n \to \mathcal{N}^m$ *be a linear transformation. Then* $\mathbf{L}$ *is uniformly differentiable on* $\mathcal{N}^n$.

**Proof.** As in the real case, $\mathbf{L}$ is differentiable with $\mathbf{DL}(\mathbf{x}) = \mathbf{L}\ \forall \mathbf{x} \in \mathcal{N}^n$. Let $\epsilon > 0$ in $\mathcal{N}$ be given. Then for any $\mathbf{s}, \mathbf{t} \in \mathcal{N}^n$ we have that

$$\begin{aligned}|\mathbf{L}(\mathbf{s}) - \mathbf{L}(\mathbf{t}) - \mathbf{DL}(\mathbf{t})(\mathbf{s}-\mathbf{t})| &= |\mathbf{L}(\mathbf{s}) - \mathbf{L}(\mathbf{t}) - \mathbf{L}(\mathbf{s}-\mathbf{t})| \\ &= |\mathbf{L}(\mathbf{s}-\mathbf{t}) - \mathbf{L}(\mathbf{s}-\mathbf{t})| = 0.\end{aligned}$$

Thus $\mathbf{L}$ is uniformly differentiable on $\mathcal{N}^n$. $\square$

**Proposition 3.8.** *Let* $\mathbf{f}, \mathbf{g}: A \to \mathcal{N}^m$ *be locally uniformly differentiable on* $A$*; and let* $\alpha \in \mathcal{N}$ *be given. Then* $\alpha\mathbf{f} + \mathbf{g}$ *is locally uniformly differentiable on* $A$*. That is, any linear combination of locally uniformly differentiable functions is again locally uniformly differentiable.*

**Proof.** If $\alpha = 0$ then there is nothing to prove; so without loss of generality we may assume $\alpha \neq 0$. Now let $\mathbf{x} \in A$ be given. Then, as in the real case, $\alpha\mathbf{f} + \mathbf{g}$ is differentiable at $\mathbf{x}$, with $\mathbf{D}(\alpha\mathbf{f} + \mathbf{g})(\mathbf{x}) = \alpha\mathbf{Df}(\mathbf{x}) + \mathbf{Dg}(\mathbf{x})$. Also $\exists \delta_{\mathbf{x}} > 0$ such that $\mathbf{f}$ and $\mathbf{g}$ are uniformly differentiable on $B_{\delta_{\mathbf{x}}}(\mathbf{x})$. Let $\epsilon > 0$ in $\mathcal{N}$ be given and let $\mathbf{s}, \mathbf{t} \in B_{\delta_{\mathbf{x}}}(\mathbf{x})$. Then $\exists \delta_{\mathbf{f}} > 0$ such that $|\mathbf{s} - \mathbf{t}| \leqslant \delta_{\mathbf{f}} \Rightarrow |\mathbf{f}(\mathbf{s}) - \mathbf{f}(\mathbf{t}) - \mathbf{Df}(\mathbf{t})(\mathbf{s}-\mathbf{t})| \leqslant \frac{\epsilon}{2|\alpha|}|\mathbf{s}-\mathbf{t}|$ and $\exists \delta_{\mathbf{g}} > 0$ such that $|\mathbf{s} - \mathbf{t}| \leqslant \delta_{\mathbf{g}} \Rightarrow |\mathbf{g}(\mathbf{s}) - \mathbf{g}(\mathbf{t}) - \mathbf{Dg}(\mathbf{t})(\mathbf{s}-\mathbf{t})| \leqslant \frac{\epsilon}{2}|\mathbf{s}-\mathbf{t}|$.

Let $\delta = \min\{\delta_{\mathbf{f}}, \delta_{\mathbf{g}}\}$. Then for $|\mathbf{s} - \mathbf{t}| \leqslant \delta$ we have that

$$\begin{aligned}&|(\alpha\mathbf{f} + \mathbf{g})(\mathbf{s}) - (\alpha\mathbf{f} + \mathbf{g})(\mathbf{t}) - \mathbf{D}(\alpha\mathbf{f} + \mathbf{g})(\mathbf{t})(\mathbf{s}-\mathbf{t})| \\ &\quad = \left|\alpha\mathbf{f}(\mathbf{s}) + \mathbf{g}(\mathbf{s}) - \big(\alpha\mathbf{f}(\mathbf{t}) + \mathbf{g}(\mathbf{t})\big) - \alpha\mathbf{Df}(\mathbf{t})(\mathbf{s}-\mathbf{t}) - \mathbf{Dg}(\mathbf{t})(\mathbf{s}-\mathbf{t})\right| \\ &\quad \leqslant |\alpha[\mathbf{f}(\mathbf{s}) - \mathbf{f}(\mathbf{t}) - \mathbf{Df}(\mathbf{t})(\mathbf{s}-\mathbf{t})]| + |\mathbf{g}(\mathbf{s}) - \mathbf{g}(\mathbf{t}) - \mathbf{Dg}(\mathbf{t})(\mathbf{s}-\mathbf{t})| \\ &\quad \leqslant |\alpha|\frac{\epsilon}{2|\alpha|}|\mathbf{s}-\mathbf{t}| + \frac{\epsilon}{2}|\mathbf{s}-\mathbf{t}| \\ &\quad = \epsilon|\mathbf{s}-\mathbf{t}|.\end{aligned}$$

$\square$

**Theorem 3.9.** *Let* $\mathbf{f}: A \to \mathcal{N}^m$ *be locally uniformly differentiable on* $A$ *and let* $\mathbf{g}: C \to \mathcal{N}^p$ *be locally uniformly differentiable on* $C$*, with* $\mathbf{f}(A) \subseteq C$*. Then* $\mathbf{g} \circ \mathbf{f}$ *is locally uniformly differentiable on* $A$.

**Proof.** Let $\mathbf{x} \in A$, let $\mathfrak{L}_{\mathbf{f}} > 0$ be an upper bound for $\{|\mathbf{Df}(\mathbf{x})(\mathbf{y})|\colon\ |\mathbf{y}| \leqslant 1\}$ and let $\mathfrak{L}_{\mathbf{g}} \geqslant 1$ be an upper bound for $\{|\mathbf{Dg}(\mathbf{f}(\mathbf{x}))(\mathbf{y})|\colon\ |\mathbf{y}| \leqslant 1\}$. Then, as in the real case, $\mathbf{D}(\mathbf{g} \circ \mathbf{f})(\mathbf{x}) = \mathbf{Dg}(\mathbf{f}(\mathbf{x})) \circ \mathbf{Df}(\mathbf{x})$. Also, $\exists \delta_1 > 0$ such that $\mathbf{f}$ is uniformly differentiable on $B_{\delta_1}(\mathbf{x})$, and $\exists \delta_2 > 0$ such that $\mathbf{g}$ is uniformly differentiable on $B_{\delta_2}(\mathbf{f}(\mathbf{x}))$. By Lemma 2.4 $\exists \delta_3 > 0$ such that whenever $\mathbf{s} \in B_{\delta_3}(\mathbf{x})$ we have that $|\mathbf{Df}(\mathbf{s})(\mathbf{y})| < 2\mathfrak{L}_{\mathbf{f}}|\mathbf{y}|$. Similarly, $\exists \delta_4 > 0$ such that whenever $\mathbf{u} \in B_{\delta_4}(\mathbf{f}(\mathbf{x}))$ we have that $|\mathbf{Dg}(\mathbf{u})(\mathbf{v})| < 2\mathfrak{L}_{\mathbf{g}}|\mathbf{v}|$. Let $\alpha = \min\{\delta_2, \delta_4\}$. Then, since $\mathbf{f}$ is continuous on $A$, $\exists \delta_5 > 0$ such that $\mathbf{f}(B_{\delta_5}(\mathbf{x})) \subseteq B_{\alpha}(\mathbf{f}(\mathbf{x}))$. Let $\delta_{\mathbf{x}} = \min\{\delta_1, \delta_3, \delta_5\}$.

Now let $\epsilon > 0$ in $\mathcal{N}$ be given. By definition, $\exists \delta_{\mathbf{g}} > 0$ in $\mathcal{N}$ such that

$$|\mathbf{g}(\mathbf{u}) - \mathbf{g}(\mathbf{v}) - \mathbf{Dg}(\mathbf{v})(\mathbf{u} - \mathbf{v})| \leqslant \frac{\epsilon}{2(\epsilon + 2\mathfrak{L}_{\mathbf{f}})} |\mathbf{u} - \mathbf{v}|$$

for all $\mathbf{u}, \mathbf{v} \in B_\alpha(\mathbf{f}(\mathbf{x}))$ satisfying $|\mathbf{u} - \mathbf{v}| < \delta_{\mathbf{g}}$. Also, $\exists \delta_{\mathbf{f}} > 0$ such that

$$|\mathbf{f}(\mathbf{s}) - \mathbf{f}(\mathbf{t}) - \mathbf{Df}(\mathbf{t})(\mathbf{s} - \mathbf{t})| \leqslant \frac{\epsilon}{4\mathfrak{L}_{\mathbf{g}}} |\mathbf{s} - \mathbf{t}|$$

for all $\mathbf{s}, \mathbf{t} \in B_{\delta_{\mathbf{x}}}(\mathbf{x})$ satisfying $|\mathbf{s} - \mathbf{t}| < \delta_{\mathbf{f}}$. Thus

$$|\mathbf{f}(\mathbf{s}) - \mathbf{f}(\mathbf{t})| \leqslant \frac{\epsilon}{4\mathfrak{L}_{\mathbf{g}}} |\mathbf{s} - \mathbf{t}| + |\mathbf{Df}(\mathbf{t})(\mathbf{s} - \mathbf{t})| \leqslant (\epsilon + 2\mathfrak{L}_{\mathbf{f}})|\mathbf{s} - \mathbf{t}|. \tag{3.2}$$

Let $\delta = \min\{\delta_{\mathbf{f}}, \frac{\delta_{\mathbf{g}}}{\epsilon + 2\mathfrak{L}_{\mathbf{f}}}\}$ and let $\mathbf{s}, \mathbf{t} \in B_{\delta_{\mathbf{x}}}(\mathbf{x})$ be such that $|\mathbf{s} - \mathbf{t}| < \delta$. Then $|\mathbf{f}(\mathbf{s}) - \mathbf{f}(\mathbf{t}) - \mathbf{Df}(\mathbf{t})(\mathbf{s} - \mathbf{t})| \leqslant \frac{\epsilon}{4\mathfrak{L}_{\mathbf{g}}} |\mathbf{s} - \mathbf{t}|$. Also, we have that $\mathbf{f}(\mathbf{s}), \mathbf{f}(\mathbf{t}) \in B_\alpha(\mathbf{f}(\mathbf{x}))$ and by (3.2), $|\mathbf{f}(\mathbf{s}) - \mathbf{f}(\mathbf{t})| < \delta_{\mathbf{g}}$. Thus,

$$\big|\mathbf{g}\big(\mathbf{f}(\mathbf{s})\big) - \mathbf{g}\big(\mathbf{f}(\mathbf{t})\big) - \mathbf{Dg}\big(\mathbf{f}(\mathbf{t})\big)\big(\mathbf{f}(\mathbf{s}) - \mathbf{f}(\mathbf{t})\big)\big| \leqslant \frac{\epsilon}{2(\epsilon + 2\mathfrak{L}_{\mathbf{f}})} |\mathbf{f}(\mathbf{s}) - \mathbf{f}(\mathbf{t})|.$$

Therefore,

$$\begin{aligned}
&\big|\mathbf{g}\big(\mathbf{f}(\mathbf{s})\big) - \mathbf{g}\big(\mathbf{f}(\mathbf{t})\big) - \mathbf{Dg}\big(\mathbf{f}(\mathbf{t})\big) \circ \mathbf{Df}(\mathbf{t})(\mathbf{s} - \mathbf{t})\big| \\
&\quad = \big|\mathbf{g}\big(\mathbf{f}(\mathbf{s})\big) - \mathbf{g}\big(\mathbf{f}(\mathbf{t})\big) - \mathbf{Dg}(\mathbf{f}(\mathbf{t}))\big(\mathbf{f}(\mathbf{s}) - \mathbf{f}(\mathbf{t})\big) \\
&\qquad + \mathbf{Dg}\big(\mathbf{f}(\mathbf{t})\big)\big(\mathbf{f}(\mathbf{s}) - \mathbf{f}(\mathbf{t})\big) - \mathbf{Dg}\big(\mathbf{f}(\mathbf{t})\big) \circ \mathbf{Df}(\mathbf{t})(\mathbf{s} - \mathbf{t})\big| \\
&\quad \leqslant \big|\mathbf{g}\big(\mathbf{f}(\mathbf{s})\big) - \mathbf{g}\big(\mathbf{f}(\mathbf{t})\big) - \mathbf{Dg}\big(\mathbf{f}(\mathbf{t})\big)\big(\mathbf{f}(\mathbf{s}) - \mathbf{f}(\mathbf{t})\big)\big| \\
&\qquad + \big|\mathbf{Dg}\big(\mathbf{f}(\mathbf{t})\big)\big(\mathbf{f}(\mathbf{s}) - \mathbf{f}(\mathbf{t}) - \mathbf{Df}(\mathbf{t})(\mathbf{s} - \mathbf{t})\big)\big| \\
&\quad \leqslant \frac{\epsilon}{2(\epsilon + 2\mathfrak{L}_{\mathbf{f}})} |\mathbf{f}(\mathbf{s}) - \mathbf{f}(\mathbf{t})| + 2\mathfrak{L}_{\mathbf{g}} |\mathbf{f}(\mathbf{s}) - \mathbf{f}(\mathbf{t}) - \mathbf{Df}(\mathbf{t})(\mathbf{s} - \mathbf{t})| \\
&\quad \leqslant \frac{\epsilon}{2} |\mathbf{s} - \mathbf{t}| + 2\mathfrak{L}_{\mathbf{g}} \frac{\epsilon}{4\mathfrak{L}_{\mathbf{g}}} |\mathbf{s} - \mathbf{t}| \\
&\quad = \frac{\epsilon}{2} |\mathbf{s} - \mathbf{t}| + \frac{\epsilon}{2} |\mathbf{s} - \mathbf{t}| \\
&\quad = \epsilon |\mathbf{s} - \mathbf{t}|.
\end{aligned}$$

□

**Lemma 3.10.** *Let $h : \mathcal{N}^2 \to \mathcal{N}$ be given by $h(x_1, x_2) = x_1 x_2$. Then $h$ is uniformly differentiable on $\mathcal{N}^2$; with $\mathbf{D}h(x_1, x_2) = (x_2 \quad x_1)$.*

**Proof.** Let $\epsilon > 0$ in $\mathcal{N}$ be given. Let $\delta = \epsilon$. Then for all $\mathbf{x} = \binom{x_1}{x_2}$ and $\mathbf{y} = \binom{y_1}{y_2}$ in $\mathcal{N}^2$ satisfying $|\mathbf{y} - \mathbf{x}| < \delta$, we have that

$$\begin{aligned}
&|h(\mathbf{y}) - h(\mathbf{x}) - (\, x_2 \quad x_1 \,)\, (\mathbf{y} - \mathbf{x})| \\
&\quad = |y_1 y_2 - x_1 x_2 - x_2(y_1 - x_1) - x_1(y_2 - x_2)| \\
&\quad = |y_1 y_2 - x_2 y_1 - x_1 y_2 + x_1 x_2| \\
&\quad = |(y_1 - x_1)(y_2 - x_2)|
\end{aligned}$$

$$
\begin{aligned}
&\leqslant |\mathbf{y}-\mathbf{x}|^2 \\
&< \epsilon|\mathbf{y}-\mathbf{x}|. \qquad \square
\end{aligned}
$$

**Proposition 3.11.** *Let $f, g : A \to \mathcal{N}$ be locally uniformly differentiable on $A$ (where $A$ is, as before, an open subset of $\mathcal{N}^n$). Then $fg$ is locally uniformly differentiable on $A$.*

**Proof.** Define $\mathbf{k} : A \to \mathcal{N}^2$ by

$$\mathbf{k}(\mathbf{x}) = \begin{pmatrix} f(\mathbf{x}) \\ g(\mathbf{x}) \end{pmatrix};$$

and let $h : \mathcal{N}^2 \to \mathcal{N}$ be as in Lemma 3.10. Then $\mathbf{k}$ is locally uniformly differentiable on $A$ and $h$ is uniformly differentiable (and hence locally uniformly differentiable) on $\mathcal{N}^2 \supseteq \mathbf{k}(A)$. It follows from Theorem 3.9 that $fg = h \circ \mathbf{k}$ is locally uniformly differentiable on $A$. $\square$

**Lemma 3.12.** *For each $j \in \{1, \ldots, n\}$, the function $f_j : \mathcal{N}^n \to \mathcal{N}$ given by $f_j(x_1, x_2, \ldots, x_n) = x_j$ is uniformly differentiable on $\mathcal{N}^n$.*

**Proof.** Let $j \in \{1, \ldots, n\}$ be given. Then for all $\mathbf{x}, \mathbf{y} \in \mathcal{N}^n$ and for all $\alpha, \beta \in \mathcal{N}$, we have that

$$f_j(\alpha\mathbf{x} + \beta\mathbf{y}) = \alpha f_j(\mathbf{x}) + \beta f_j(\mathbf{y}).$$

Hence $f_j$ is a linear transformation from $\mathcal{N}^n$ to $\mathcal{N}$. It follows from Proposition 3.7 that $f_j$ is uniformly differentiable on $\mathcal{N}^n$. $\square$

Using the results of Proposition 3.11 and Lemma 3.12, we infer that any monomial function is locally uniformly differentiable on $\mathcal{N}^n$. It then follows from Proposition 3.8 that any polynomial function is locally uniformly differentiable on $\mathcal{N}^n$.

**Corollary 3.13.** *Let $f : \mathcal{N}^n \to \mathcal{N}$ be a polynomial function. Then $f$ is locally uniformly differentiable on $\mathcal{N}^n$.*

**Corollary 3.14.** *Let $\mathbf{f} : \mathcal{N}^n \to \mathcal{N}^m$ be given by*

$$\mathbf{f} = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_m \end{pmatrix}$$

*with $f_i$ a polynomial function from $\mathcal{N}^n$ to $\mathcal{N}$ for all $i \in \{1, \ldots, m\}$. Then $\mathbf{f}$ is locally uniformly differentiable on $\mathcal{N}^n$.*

We start this section with some preliminary results needed to prove the inverse function theorem. Let $\delta_1 > 0$ and let $\boldsymbol{\phi} : B_{\delta_1}(\mathbf{0}) \subset \mathcal{N}^n \to \mathcal{N}^n$ be such that

$$|\boldsymbol{\phi}(\mathbf{t})| \leqslant c|\mathbf{t}| \tag{4.1}$$

for every $\mathbf{t} \in B_{\delta_1}(\mathbf{0})$ where $0 < c \ll 1$. Then $\boldsymbol{\phi}(B_{\delta_1}(\mathbf{0})) \subseteq B_{\delta_1}(\mathbf{0})$. For $m \in \mathbb{N}$ let $\boldsymbol{\phi}^{[m]} = \underbrace{\boldsymbol{\phi} \circ \cdots \circ \boldsymbol{\phi}}_{m \text{ times}}$ and set $\boldsymbol{\phi}^{[0]} = \mathbf{I}$ (the identity map on $\mathcal{N}^n$). Using induction, it can be shown that $\forall m \in \mathbb{N}$ we have that

(a) $\boldsymbol{\phi}^{[m]}(B_{\delta_1}(\mathbf{0})) \subseteq B_{\delta_1}(\mathbf{0})$; and
(b) $|\boldsymbol{\phi}^{[m]}(\mathbf{t})| \leqslant c^m |\mathbf{t}|$.

**Lemma 4.1.** *Let $\boldsymbol{\phi} : B_{\delta_1}(\mathbf{0}) \to \mathcal{N}^n$ be continuous on $B_{\delta_1}(\mathbf{0})$ and satisfy* (4.1)*. Let $\delta \leqslant (1-c)\delta_1$ and let $\boldsymbol{\psi}(\mathbf{t}) = \sum_{m=0}^{\infty} \boldsymbol{\phi}^{[m]}(\mathbf{t})$, $\forall \mathbf{t} \in B_\delta(\mathbf{0})$. Then:*

- $|\boldsymbol{\psi}(\mathbf{t})| \leqslant \frac{|\mathbf{t}|}{1-c}$*; and*
- $\boldsymbol{\psi}(\mathbf{t}) - \boldsymbol{\phi}[\boldsymbol{\psi}(\mathbf{t})] = \mathbf{t}$.

**Proof.** By (b) above, we have that $|\boldsymbol{\phi}^{[m]}(\mathbf{t})| \leqslant c^m|\mathbf{t}|$. Also, we have that $\lim_{m\to\infty} c^m = 0$ since $c \ll 1$ and the skeleton group of $\mathcal{N}$ is Archimedean. Thus, $\lim_{m\to\infty} |\boldsymbol{\phi}^{[m]}(\mathbf{t})| = 0$ and so $\sum_{m=0}^{\infty} \boldsymbol{\phi}^{[m]}(\mathbf{t})$ converges. For each $r \in \mathbb{N}$, let $\boldsymbol{\psi}_r(\mathbf{t}) = \sum_{m=0}^{r} \boldsymbol{\phi}^{[m]}(\mathbf{t})$. Then

$$|\boldsymbol{\psi}_r(\mathbf{t})|, |\boldsymbol{\psi}(\mathbf{t})| \leqslant \sum_{m=0}^{\infty} \left|\boldsymbol{\phi}^{[m]}(\mathbf{t})\right| \leqslant |\mathbf{t}| \sum_{m=0}^{\infty} c^m = \frac{|\mathbf{t}|}{1-c}.$$

Therefore, $\boldsymbol{\psi}_r(\mathbf{t}), \boldsymbol{\psi}(\mathbf{t}) \in B_{\delta_1}(\mathbf{0})$ $\forall \mathbf{t} \in B_\delta(\mathbf{0})$ and $\forall r \in \mathbb{N}$. Furthermore,

$$\boldsymbol{\psi}_r - \boldsymbol{\phi} \circ \boldsymbol{\psi}_r = \sum_{m=0}^{r} \boldsymbol{\phi}^{[m]} - \sum_{m=1}^{r+1} \boldsymbol{\phi}^{[m]} = \mathbf{I} - \boldsymbol{\phi}^{[r+1]};$$

and hence

$$\boldsymbol{\psi}_r(\mathbf{t}) - \boldsymbol{\phi}[\boldsymbol{\psi}_r(\mathbf{t})] = \mathbf{t} - \boldsymbol{\phi}^{[r+1]}(\mathbf{t}). \tag{4.2}$$

It is readily seen that $\lim_{r\to\infty} \boldsymbol{\psi}_r(\mathbf{t}) = \boldsymbol{\psi}(\mathbf{t})$, and so $\lim_{r\to\infty} \boldsymbol{\phi}[\boldsymbol{\psi}_r(\mathbf{t})] = \boldsymbol{\phi}[\boldsymbol{\psi}(\mathbf{t})]$ since $\boldsymbol{\phi}$ is continuous. Also, $\lim_{r\to\infty} \boldsymbol{\phi}^{[r]}(\mathbf{t}) = \mathbf{0}$. Thus, by letting $r \to \infty$ on both sides of (4.2), we obtain:

$$\boldsymbol{\psi}(\mathbf{t}) - \boldsymbol{\phi}[\boldsymbol{\psi}(\mathbf{t})] = \mathbf{t}.$$ □

**Lemma 4.2.** *Let $\mathbf{g} : A \to \mathcal{N}^n$ be locally uniformly differentiable on $A$, with $J\mathbf{g}(\mathbf{t}_1) \neq 0$, where $J\mathbf{g}(\mathbf{t}_1)$ denotes the Jacobian (determinant) of $\mathbf{g}$ at $\mathbf{t}_1$. Then $\exists \delta, \eta > 0$ and a function $\mathbf{F}$ defined on $B_\eta(\mathbf{x}_1)$ where $\mathbf{x}_1 = \mathbf{g}(\mathbf{t}_1)$ such that:*

(i) $B_\delta(\mathbf{t}_1) \subseteq A$;

(ii) $\mathbf{g}|_{B_\delta(\mathbf{t}_1)}$ *is injective;*

(iii) $B_\eta(\mathbf{x}_1) \subseteq \mathbf{g}(B_\delta(\mathbf{t}_1))$ *and* $\mathbf{F}(B_\eta(\mathbf{x}_1)) \subseteq B_\delta(\mathbf{t}_1)$;

(iv) $\mathbf{g}[\mathbf{F}(\mathbf{x})] = \mathbf{x}\ \forall \mathbf{x} \in B_\eta(\mathbf{x}_1)$; *and*

(v) $\mathbf{F}$ *is uniformly differentiable on* $B_\eta(\mathbf{x}_1)$ *with* $\mathbf{DF}(\mathbf{x}) = [\mathbf{Dg}(\mathbf{t})]^{-1}$ *where* $\mathbf{x} = \mathbf{g}(\mathbf{t})$ *and* $\mathbf{t} \in B_\delta(\mathbf{t}_1)$.

**Proof.** Without loss of generality, we may assume that $\mathbf{t}_1 = \mathbf{0}$ and $\mathbf{x}_1 = \mathbf{0}$; for if this is not the case then we can replace $\mathbf{g}(\mathbf{t})$ with $\tilde{\mathbf{g}}(\mathbf{t}) := \mathbf{g}(\mathbf{t}+\mathbf{t}_1) - \mathbf{x}_1$. Since $\mathbf{g}$ is locally uniformly differentiable, $\exists \omega_0 > 0$ such that $\mathbf{g}$ is uniformly differentiable on $B_{\omega_0}(\mathbf{0})$. Also, since $\mathbf{g}$ is $C^1$, $\exists \omega_1$ such that $J\mathbf{g}(\mathbf{t}) \neq 0\ \forall \mathbf{t} \in B_{\omega_1}(\mathbf{0})$. Let $\omega = \min\{\omega_0, \omega_1\}$. By Lemma 3.6, $\mathbf{g}$ satisfies (3.1). Let $\mathbf{L} = \mathbf{Dg}(\mathbf{0})$; then $\mathbf{L}^{-1}$ exists since $J\mathbf{g}(\mathbf{0}) \neq 0$. Let $\boldsymbol{\phi} = \mathbf{I} - \mathbf{L}^{-1} \circ \mathbf{g}$. It follows that $\boldsymbol{\phi}(\mathbf{0}) = \mathbf{0}$ and

$$\mathbf{D}\boldsymbol{\phi}(\mathbf{0}) = \mathbf{D}\big(\mathbf{I} - \mathbf{L}^{-1} \circ \mathbf{g}\big)(\mathbf{0}) = \mathbf{I} - \mathbf{L}^{-1} \circ \mathbf{Dg}(\mathbf{0}) = \mathbf{I} - \mathbf{L}^{-1} \circ \mathbf{L} = \mathbf{0}.$$

Let $c \in \mathcal{N}$ be such that $0 < c \ll 1$. Since $\boldsymbol{\phi}$ satisfies (3.1) at $\mathbf{0}$, $\exists \delta_0 > 0$ such that $\forall \mathbf{s}, \mathbf{t} \in B_{\delta_0}(\mathbf{0})$ we have that $|\boldsymbol{\phi}(\mathbf{s}) - \boldsymbol{\phi}(\mathbf{t}) - \mathbf{D}\boldsymbol{\phi}(\mathbf{0})(\mathbf{s} - \mathbf{t})| < c|\mathbf{s} - \mathbf{t}|$. Since $A$ is open, we may choose $\delta_0$ small enough so $B_{\delta_0}(\mathbf{0}) \subset A$. Thus,

$$|\boldsymbol{\phi}(\mathbf{s}) - \boldsymbol{\phi}(\mathbf{t})| < c|\mathbf{s} - \mathbf{t}|. \tag{4.3}$$

Let $\mathbf{s}, \mathbf{t} \in B_{\delta_0}(\mathbf{0})$ be such that $\mathbf{g}(\mathbf{s}) = \mathbf{g}(\mathbf{t})$. Then $\boldsymbol{\phi}(\mathbf{s}) - \boldsymbol{\phi}(\mathbf{t}) = \mathbf{s} - \mathbf{t}$ and hence $|\mathbf{s} - \mathbf{t}| = |\boldsymbol{\phi}(\mathbf{s}) - \boldsymbol{\phi}(\mathbf{t})| \leqslant c|\mathbf{s} - \mathbf{t}|$ (by (4.3)). Since $c$ is infinitely small, it follows that $\mathbf{s} = \mathbf{t}$ and hence $\mathbf{g}|_{B_{\delta_0}(\mathbf{0})}$ is one-to-one. Let $\bar{\mathfrak{L}}$ be an upper bound for $\{|\mathbf{L}^{-1}(\mathbf{t})| \colon |\mathbf{t}| \leqslant 1\}$. Since $\mathbf{g}$ satisfies (3.1) at $\mathbf{0}$, $\exists \delta_g > 0$ such that $\forall \mathbf{s}, \mathbf{t} \in B_{\delta_g}(\mathbf{0})$ we have that

$$|\mathbf{g}(\mathbf{s}) - \mathbf{g}(\mathbf{t}) - \mathbf{L}(\mathbf{s} - \mathbf{t})| < \frac{1}{2\bar{\mathfrak{L}}}|\mathbf{s} - \mathbf{t}|. \tag{4.4}$$

Also, since $\mathbf{g}$ is $C^1$, it follows from Lemma (2.4) that $\exists \delta_d > 0$ such that $\forall \mathbf{s} \in B_{\delta_d}(\mathbf{0})$ and $\forall \mathbf{x} \in \mathcal{N}^n$ we have that $|(\mathbf{Dg}(\mathbf{s}))^{-1}\mathbf{x}| < 2\bar{\mathfrak{L}}|\mathbf{x}|$. Let $\delta = \min\{(1-c)\delta_0, \omega, \delta_g, \delta_d\}$. Then $B_\delta(\mathbf{0}) \subseteq B_{\delta_0}(\mathbf{0}) \subset A$ and hence $\mathbf{g}|_{B_\delta(\mathbf{0})}$ is one-to-one. This shows (i) and (ii).

By (4.3) with $\mathbf{t} = \mathbf{0}$ we have that $|\boldsymbol{\phi}(\mathbf{s})| < c|\mathbf{s}|\ \forall \mathbf{s} \in B_\delta(\mathbf{0})$, and so we have a function $\boldsymbol{\psi}$ with the properties of Lemma 4.1. Let $\eta = \frac{\delta}{\bar{\mathfrak{L}}}(1-c)$ and define $\mathbf{F}(\mathbf{x}) = \boldsymbol{\psi}(\mathbf{L}^{-1}(\mathbf{x}))$ for all $\mathbf{x} \in B_\eta(\mathbf{0})$. Thus, $\forall \mathbf{x} \in B_\eta(\mathbf{0})$, we have that

$$|\mathbf{F}(\mathbf{x})| = \big|\boldsymbol{\psi}\big(\mathbf{L}^{-1}(\mathbf{x})\big)\big| \leqslant \frac{|\mathbf{L}^{-1}(\mathbf{x})|}{(1-c)} \leqslant \frac{\bar{\mathfrak{L}}|\mathbf{x}|}{(1-c)} \leqslant \frac{\bar{\mathfrak{L}}\eta}{(1-c)} = \delta.$$

Hence $\mathbf{F}(B_\eta(\mathbf{0})) \subseteq B_\delta(\mathbf{0})$. Furthermore, $(\mathbf{I} - \boldsymbol{\phi})|_{B_\delta(\mathbf{0})} = (\mathbf{L}^{-1} \circ \mathbf{g})|_{B_\delta(\mathbf{0})}$; and by Lemma 4.1

$$\big((\mathbf{I} - \boldsymbol{\phi}) \circ \boldsymbol{\psi}\big)|_{B_\delta(\mathbf{0})} = \mathbf{I}|_{B_\delta(\mathbf{0})}.$$

Thus,

$$\big(\mathbf{L}^{-1} \circ \mathbf{g} \circ \boldsymbol{\psi}\big)|_{B_\delta(\mathbf{0})} = \mathbf{I}|_{B_\delta(\mathbf{0})};$$

and hence

$$\mathbf{g}\big(\boldsymbol{\psi}(\mathbf{t})\big) = \mathbf{L}(\mathbf{t}) \quad \forall \mathbf{t} \in B_\delta(\mathbf{0}).$$

Let $\mathbf{x} \in B_\eta(\mathbf{0})$ and set $\mathbf{t} = \mathbf{L}^{-1}(\mathbf{x})$. Then $|\mathbf{t}| \leqslant \bar{\mathfrak{L}}|\mathbf{x}| \leqslant (1-c)\delta < \delta$. Thus, $\mathbf{L}^{-1}(\mathbf{x}) \in B_\delta(\mathbf{0})$. It follows that

$$\mathbf{g}\big(\mathbf{F}(\mathbf{x})\big) = \mathbf{g}\big(\boldsymbol{\psi}\big(\mathbf{L}^{-1}(\mathbf{x})\big)\big) = \mathbf{L}\big(\mathbf{L}^{-1}(\mathbf{x})\big) = \mathbf{x} \quad \forall \mathbf{x} \in B_\eta(\mathbf{0})$$

and hence $B_\eta(\mathbf{0}) \subseteq \mathbf{g}(B_\delta(\mathbf{0}))$, since $\mathbf{F}(\mathbf{x}) \in B_\delta(\mathbf{0})$ $\forall \mathbf{x} \in B_\eta(\mathbf{0})$. This shows (iii) and (iv).

**Claim.** $|\mathbf{s}-\mathbf{t}| < 2\bar{\mathfrak{L}}|\mathbf{g}(\mathbf{s}) - \mathbf{g}(\mathbf{t})|$ $\forall \mathbf{s}, \mathbf{t} \in B_\delta(\mathbf{0})$.

Let $\mathbf{s}, \mathbf{t} \in B_\delta(\mathbf{0})$. Then by (4.4),

$$|\mathbf{g}(\mathbf{s}) - \mathbf{g}(\mathbf{t}) - \mathbf{L}(\mathbf{s}-\mathbf{t})| < \frac{|\mathbf{s}-\mathbf{t}|}{2\bar{\mathfrak{L}}}.$$

It follows that

$$|\mathbf{g}(\mathbf{s}) - \mathbf{g}(\mathbf{t})| > |\mathbf{L}(\mathbf{s}-\mathbf{t})| - \frac{|\mathbf{s}-\mathbf{t}|}{2\bar{\mathfrak{L}}} \geqslant \frac{|\mathbf{s}-\mathbf{t}|}{\bar{\mathfrak{L}}} - \frac{|\mathbf{s}-\mathbf{t}|}{2\bar{\mathfrak{L}}} = \frac{|\mathbf{s}-\mathbf{t}|}{2\bar{\mathfrak{L}}}.$$

This completes the proof of the claim.

Let $\epsilon > 0$ in $\mathcal{N}$ be given. Since $\mathbf{g}|_{B_\delta(\mathbf{0})}$ is uniformly differentiable, there exists $\delta_1 > 0$ in $\mathcal{N}$ such that whenever $|\mathbf{s}-\mathbf{t}| < \delta_1$

$$|\mathbf{g}(\mathbf{s}) - \mathbf{g}(\mathbf{t}) - \mathbf{Dg}(\mathbf{t})(\mathbf{s}-\mathbf{t})| < \frac{\epsilon|\mathbf{s}-\mathbf{t}|}{4\bar{\mathfrak{L}}^2}.$$

Let $\xi = \frac{\delta_1}{2\bar{\mathfrak{L}}}$ and let $\mathbf{x}, \mathbf{y} \in B_\eta(\mathbf{0})$ with $|\mathbf{x}-\mathbf{y}| < \xi$. Since $B_\eta(\mathbf{0}) \subseteq \mathbf{g}(B_\delta(\mathbf{0}))$, then $\exists \mathbf{t_x}, \mathbf{t_y} \in B_\delta(\mathbf{0})$ such that $\mathbf{g}(\mathbf{t_x}) = \mathbf{x}$, and $\mathbf{g}(\mathbf{t_y}) = \mathbf{y}$. Since $\mathbf{F}(B_\eta(\mathbf{0})) \subseteq B_\delta(\mathbf{0})$ we get that $\mathbf{F}(\mathbf{g}(\mathbf{t_x})) = \mathbf{F}(\mathbf{x}) \in B_\delta(\mathbf{0})$. Thus, $\mathbf{g}(\mathbf{F}(\mathbf{x})) = \mathbf{g}(\mathbf{F}(\mathbf{g}(\mathbf{t}_x))) = \mathbf{g}(\mathbf{t_x})$ by (iv). Since $\mathbf{g}$ is one-to-one on $B_\delta(\mathbf{0})$, it follows that $\mathbf{F}(\mathbf{x}) = \mathbf{t_x}$. Similarly $\mathbf{F}(\mathbf{y}) = \mathbf{t_y}$. Moreover, we have that

$$|\mathbf{t_y} - \mathbf{t_x}| < 2\bar{\mathfrak{L}}|\mathbf{g}(\mathbf{t_y}) - \mathbf{g}(\mathbf{t_x})| = 2\bar{\mathfrak{L}}|\mathbf{y}-\mathbf{x}| < 2\bar{\mathfrak{L}}\xi = \delta_1.$$

Note: $(\mathbf{Dg}(\mathbf{t}))^{-1}$ exists since $J\mathbf{g}(\mathbf{t}) \neq 0$ $\forall \mathbf{t} \in B_\omega(\mathbf{0}) \supseteq B_\delta(\mathbf{0})$. Now,

$$\begin{aligned}
&\left|\mathbf{F}(\mathbf{y}) - \mathbf{F}(\mathbf{x}) - \big(\mathbf{Dg}(\mathbf{t_x})\big)^{-1}(\mathbf{y}-\mathbf{x})\right| \\
&\quad = \left|\big(\mathbf{Dg}(\mathbf{t_x})\big)^{-1}\big(\mathbf{y}-\mathbf{x}-\mathbf{Dg}(\mathbf{t_x})\big(\mathbf{F}(\mathbf{y})-\mathbf{F}(\mathbf{x})\big)\big)\right| \\
&\quad < 2\bar{\mathfrak{L}}\left|\mathbf{y}-\mathbf{x}-\mathbf{Dg}(\mathbf{t}_x)\big(\mathbf{F}(\mathbf{y})-\mathbf{F}(\mathbf{x})\big)\right| \\
&\quad = 2\bar{\mathfrak{L}}|\mathbf{g}(\mathbf{t_y}) - \mathbf{g}(\mathbf{t_x}) - \mathbf{Dg}(\mathbf{t_x})(\mathbf{t_y}-\mathbf{t_x})|
\end{aligned}$$

$$
\begin{aligned}
&< 2\bar{\mathfrak{L}}\left(\frac{\epsilon}{4\bar{\mathfrak{L}}^2}\right)|\mathbf{t_y} - \mathbf{t_x}| = \frac{\epsilon|\mathbf{t_y} - \mathbf{t_x}|}{2\bar{\mathfrak{L}}} \\
&< \frac{\epsilon}{2\bar{\mathfrak{L}}}(2\bar{\mathfrak{L}})|\mathbf{g}(\mathbf{t_y}) - \mathbf{g}(\mathbf{t_x})| \\
&= \epsilon|\mathbf{y} - \mathbf{x}|.
\end{aligned}
$$

Thus $\mathbf{F}$ is uniformly differentiable on $B_\eta(\mathbf{0})$, and $\mathbf{DF}(\mathbf{x}) = (\mathbf{Dg}(\mathbf{t}))^{-1}$ where $\mathbf{g}(\mathbf{t}) = \mathbf{x}$. □

**Theorem 4.3** (Inverse Function Theorem). *Let* $\mathbf{g}: A \to \mathcal{N}^n$ *be locally uniformly differentiable on the open set* $A$. *If* $\mathbf{t}_0 \in A$ *is such that* $J\mathbf{g}(\mathbf{t}_0) \neq 0$ *then there is a neighborhood* $\Omega$ *of* $t_0$ *such that:*

(i) $\mathbf{g}|_\Omega$ *is injective;*
(ii) $\mathbf{g}(\Omega)$ *is open;*
(iii) *the inverse* $\mathbf{f}$ *of* $\mathbf{g}|_\Omega$ *is locally uniformly differentiable on* $\mathbf{g}(\Omega)$*; and* $\mathbf{Df}(\mathbf{x}) = [\mathbf{Dg}(\mathbf{t})]^{-1}$ *for* $\mathbf{t} \in \Omega$ *and* $\mathbf{x} = \mathbf{g}(\mathbf{t})$.

**Proof.** Using Lemma 4.2, we can find a neighborhood $\Omega_0$ of $\mathbf{t}_0$ such that $\mathbf{g}|_{\Omega_0}$ is one-to-one. Also, since $\mathbf{g}$ is $C^1$ and $J\mathbf{g}(\mathbf{t}_0) \neq 0$, there exists a neighborhood $\Omega_1$ of $\mathbf{t}_0$ such that $J\mathbf{g}(\mathbf{t}) \neq 0$ $\forall \mathbf{t} \in \Omega_1$. Let $\Omega \subseteq \Omega_0 \cap \Omega_1$ be a neighborhood of $\mathbf{t}_0$. Then $\mathbf{g}|_\Omega$ is one-to-one. Let $\mathbf{f} = (\mathbf{g}|_\Omega)^{-1}$ with domain $\mathbf{g}(\Omega)$. Let $\mathbf{t} \in \Omega$ and $\mathbf{x} = \mathbf{g}(\mathbf{t})$. Lemma 4.2 applied to $\mathbf{g}|_\Omega$ at point $\mathbf{t}$ gives us $\delta$, $\eta$, and $\mathbf{F}$ as stated in the lemma. Since $B_\eta(\mathbf{x}) \subseteq \mathbf{g}(B_\delta(\mathbf{t})) \subseteq \mathbf{g}(\Omega)$ and $\mathbf{g}$ is one-to-one on $\Omega$, it follows that

$$\mathbf{g}\big(\mathbf{F}(\mathbf{y})\big) = \mathbf{y} = \mathbf{g}\big(\mathbf{f}(\mathbf{y})\big)$$

and hence

$$\mathbf{F}(\mathbf{y}) = \mathbf{f}(\mathbf{y}) \quad \forall \mathbf{y} \in B_\eta(\mathbf{x}).$$

Since each $\mathbf{x} \in \mathbf{g}(\Omega)$ has such a neighborhood $B_\eta(\mathbf{x})$, $\mathbf{g}(\Omega)$ is open. Since $\forall \mathbf{x} \in \mathbf{g}(\Omega)$ $\mathbf{F}$ is uniformly differentiable on $B_\eta(\mathbf{x})$, $\mathbf{f}$ is locally uniformly differentiable. Finally, $\mathbf{Df}(\mathbf{x}) = \mathbf{DF}(\mathbf{x}) = [\mathbf{Dg}(\mathbf{t})]^{-1}$. □

As in the real case, the inverse function theorem will be used to prove the implicit function theorem.

## 5. IMPLICIT FUNCTION THEOREM

Let $\mathbf{\Phi}: \mathcal{N}^n \to \mathcal{N}^m$ be a $C^1$ function. Denote

$$\tilde{J}\mathbf{\Phi}(\mathbf{x}) = \det\left(\frac{\partial(\Phi_1, \ldots, \Phi_m)}{\partial(x_{n-m+1}, \ldots, x_n)}\right)$$

and

$$\hat{\mathbf{x}} = (x_1, \ldots, x_{n-m}).$$

**Theorem 5.1.** *Let $\mathbf{\Phi}: A \to \mathcal{N}^m$ be locally uniformly differentiable, where $A \subseteq \mathcal{N}^n$ is open and $1 \leqslant m < n$. Let $\mathbf{t}_0 \in A$ be such that $\mathbf{\Phi}(\mathbf{t}_0) = \mathbf{0}$ and $\tilde{J}\mathbf{\Phi}(\mathbf{t}_0) \neq 0$. Then there exist a neighborhood $U$ of $\mathbf{t}_0$, a neighborhood $R$ of $\hat{\mathbf{t}}_0$ and $\boldsymbol{\phi}: R \to \mathcal{N}^m$ locally uniformly differentiable such that for every $\mathbf{t} \in U$:*

$$\tilde{J}\mathbf{\Phi}(\mathbf{t}) \neq 0$$

*and*

$$\{\mathbf{t} \in U: \ \mathbf{\Phi}(\mathbf{t}) = \mathbf{0}\} = \left\{\left(\hat{\mathbf{t}}, \boldsymbol{\phi}(\hat{\mathbf{t}})\right): \ \hat{\mathbf{t}} \in R\right\}.$$

**Proof.** Since $\mathbf{\Phi}$ is $C^1$ and $\tilde{J}\mathbf{\Phi}(\mathbf{t}_0) \neq 0$, $\exists U_0$ neighborhood of $\mathbf{t}_0$ such that $\tilde{J}\mathbf{\Phi}(\mathbf{t}) \neq 0$ $\forall \mathbf{t} \in U_0$. Let $\mathbf{g}: U_0 \to \mathcal{N}^n$ be defined as

$$\begin{aligned} g_i(\mathbf{t}) &= t_i, && 1 \leqslant i \leqslant n-m \\ g_{n-m+j}(\mathbf{t}) &= \Phi_j(\mathbf{t}), && 1 \leqslant j \leqslant m. \end{aligned}$$

Then $\mathbf{g}$ is locally uniformly differentiable and has the Jacobian matrix

$$\left(\begin{array}{ccc|ccc} 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 & \cdots & 0 \\ \hline \Phi_1^1(\mathbf{t}) & \cdots & \Phi_{n-m}^1(\mathbf{t}) & \Phi_{n-m+1}^1(\mathbf{t}) & \cdots & \Phi_n^1(\mathbf{t}) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \Phi_1^m(\mathbf{t}) & \cdots & \Phi_{n-m}^m(\mathbf{t}) & \Phi_{n-m+1}^m(\mathbf{t}) & \cdots & \Phi_n^m(\mathbf{t}) \end{array}\right).$$

Thus, $J\mathbf{g}(\mathbf{t}) = \tilde{J}\mathbf{\Phi}(\mathbf{t}) \neq 0$ on $U_0$. Applying the Inverse Function Theorem (Theorem 4.3) to $\mathbf{g}$ at $\mathbf{t}_0$, we get a neighborhood $U \subseteq U_0$ such that $\mathbf{g}(U)$ is open, and $\mathbf{g}|_U$ is one-to-one. Additionally, $\mathbf{g}|_U$ has an inverse $\mathbf{f}$ which is also locally uniformly differentiable. Let $R = \{\hat{\mathbf{t}}: \ (\hat{\mathbf{t}}, \mathbf{0}) \in \mathbf{g}(U)\}$. Then $R$ is open since $\mathbf{g}(U)$ is open. Let $\boldsymbol{\phi}: R \to \mathcal{N}^m$ be defined as

$$\phi_l(\hat{\mathbf{t}}) = f_{n-m+l}(\hat{\mathbf{t}}, \mathbf{0}) \quad 1 \leqslant l \leqslant m.$$

Then,

$$\mathbf{t} \in U \quad \text{and} \quad \mathbf{\Phi}(\mathbf{t}) = \mathbf{0} \quad \Longleftrightarrow \quad \hat{\mathbf{t}} \in R \quad \text{and} \quad \mathbf{g}(\mathbf{t}) = (\hat{\mathbf{t}}, \mathbf{0}).$$

Moreover, since $\mathbf{g}|_U$ and $\mathbf{f}$ are inverses, it follows that

$$\mathbf{g}(\mathbf{t}) = (\hat{\mathbf{t}}, \mathbf{0}) \quad \Longleftrightarrow \quad \mathbf{t} = \mathbf{f}(\hat{\mathbf{t}}, \mathbf{0}).$$

Thus,

$$\begin{aligned}\{\mathbf{t}\in U:\ \boldsymbol{\Phi}(\mathbf{t})=\mathbf{0}\} &= \{\mathbf{t}\in U:\ \mathbf{g}(\mathbf{t})=(\hat{\mathbf{t}},\mathbf{0}),\hat{\mathbf{t}}\in R\}\\ &= \{\mathbf{t}\in U:\ \mathbf{t}=\mathbf{f}(\hat{\mathbf{t}},\mathbf{0}),\hat{\mathbf{t}}\in R\}\\ &= \left\{\left(\hat{\mathbf{t}},\boldsymbol{\phi}(\hat{\mathbf{t}})\right):\ \hat{\mathbf{t}}\in R\right\}.\end{aligned}$$

□

REFERENCES


[1] Berz M. – Calculus numerics on Levi-Civita fields, in: M. Berz, C. Bischof, G. Corliss, A. Griewank (Eds.), Computational Differentiation: Techniques, Applications, and Tools, SIAM, Philadelphia, 1996, pp. 19–35.

[2] Hahn H. – Über die nichtarchimedischen Gröensysteme, Sitzungsbericht der Wiener Akademie der Wissenschaften Abt. 2a **117** (1907) 601–655.

[3] Krull W. – Allgemeine Bewertungstheorie, J. Reine Angew. Math. **167** (1932) 160–196.

[4] Levi-Civita T. – Sugli infiniti ed infinitesimi attuali quali elementi analitici, Atti Ist. Veneto di Sc., Lett. ed Art., 7a **4** (1892) 1765.

[5] Levi-Civita T. – Sui numeri transfiniti, Rend. Acc. Lincei, 5a **7** (91) (1898) 113.

[6] MacLane S. – The universality of formal power series fields, Bulletin American Mathematical Society **45** (1939) 888.

[7] Priess-Crampe S. – Angeordnete Strukturen: Gruppen, Körper, projektive Ebenen, Springer, Berlin, 1983.

[8] Rayner F.J. – Algebraically closed fields analogous to fields of Puiseux series, J. London Math. Soc. **8** (1974) 504–506.

[9] Ribenboim P. – Fields: Algebraically closed and others, Manuscripta Mathematica **75** (1992) 115–150.

[10] Shamseddine K. – New elements of analysis on the Levi-Civita field, PhD thesis, Michigan State University, East Lansing, MI, USA, 1999. Also Michigan State University Report MSUCL-1147.

[11] Shamseddine K. – On the topological structure of the Levi-Civita field, J. Math. Anal. Appl. **368** (2010) 281–292.

[12] Shamseddine K., Berz M. – Exception handling in derivative computation with non-Archimedean calculus, in: M. Berz, C. Bischof, G. Corliss, A. Griewank (Eds.), Computational Differentiation: Techniques, Applications, and Tools, SIAM, Philadelphia, 1996, pp. 37–51.

[13] Shamseddine K., Berz M. – Convergence on the Levi-Civita field and study of power series, in: Proc. Sixth International Conference on Nonarchimedean Analysis, Marcel Dekker, New York, NY, 2000, pp. 283–299.

[14] Shamseddine K., Berz M. – The differential algebraic structure of the Levi-Civita field and applications, International Journal of Applied Mathematics **3** (2000) 449–465.

[15] Shamseddine K., Berz M. – Intermediate values inverse functions on non-Archimedean fields, International Journal of Mathematics and Mathematical Sciences **30** (2002) 165–176.

[16] Shamseddine K., Berz M. – Measure theory integration on the Levi-Civita field, Contemporary Mathematics **319** (2003) 369–387.

[17] Shamseddine K., Berz M. – Analytical properties of power series on Levi-Civita fields, Annales Mathématiques Blaise Pascal **12** (2) (2005) 309–329.

[18] Shamseddine K., Berz M. – Generalized power series on a non-Archimedean field, Indag. Mathem. **17** (3) (2006) 457–477.

[19] Shamseddine K., Berz M. – Intermediate value theorem for analytic functions on a Levi-Civita field, Bulletin of the Belgian Mathematical Society – Simon Stevin **14** (2007) 1001–1015.

[20] Shamseddine K., Zeidan V. – One-dimensional optimization on non-Archimedean fields, Journal of Nonlinear and Convex Analysis **2** (2001) 351–361.

[21] Shamseddine K., Zeidan V. – Constrained second order optimization on non-Archimedean fields, Indag. Mathem. **14** (2003) 81–101.


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