ON THE COMPLEX LEVI-CIVITA FIELD: ALGEBRAIC AND TOPOLOGICAL STRUCTURES, AND FOUNDATIONS FOR ANALYSIS

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Abstract. In this paper, we will introduce the complex Levi-Civita field $\mathcal{C}$. We start by reviewing the algebraic structure of the field; in particular, $\mathcal{C}$ is the smallest non-Archimedean valued field extension of the complex numbers field $\mathbb{C}$ that is algebraically closed and complete in the valuation topology.

Two topologies on $\mathcal{C}$ will be studied in detail: the valuation topology induced by a non-Archimedean valuation on the field, and another weaker topology induced by a family of seminorms, which we will call weak topology. We show that each of the two topologies results from a metric on $\mathcal{C}$ and that the valuation topology is not a vector topology while the weak topology is. Then we give simple characterizations of open, closed, and compact sets in both topologies.

Finally, we define continuity and differentiability for a $\mathcal{C}$-valued function at a point or on a subset of $\mathcal{C}$, we present key results for such functions, and we set the foundations for a Cauchy-like analysis theory on the field $\mathcal{C}$.

1. Introduction

In this section we introduce the Levi-Civita field $\mathcal{R}$ and its complex counterpart $\mathcal{C}$, and we briefly review their algebraic properties. We recall that the elements of $\mathcal{R}$ and $\mathcal{C}$ are functions from $\mathbb{Q}$ to $\mathbb{R}$ and $\mathbb{C}$, respectively, with left-finite support (denoted by supp). That is, below every rational number q, there are only finitely many points where the given function does not vanish. For the further discussion, it is convenient to introduce the following terminology.

Definition 1.1. (λ, $\sim$, $\approx$, $=_q$) For $x \neq 0$ in $\mathcal{R}$ or $\mathcal{C}$, we let $\lambda(x) = \min(\text{supp}(x))$, which exists because of the left-finiteness of $\text{supp}(x)$; and we let $\lambda(0) = +\infty$. Moreover, we denote the value of x at $q \in \mathbb{Q}$ with brackets like $x[q]$.

Given $x, y \neq 0$ in $\mathcal{R}$ or $\mathcal{C}$, we say $x \sim y$ if $\lambda(x) = \lambda(y)$; and we say $x \approx y$ if $\lambda(x) = \lambda(y)$ and $x[\lambda(x)] = y[\lambda(y)]$. Finally, for any $q \in \mathbb{Q}$, we say $x =_q y$ if $x[p] = y[p]$ for all $p \leq q$ in $\mathbb{Q}$.

At this point, these definitions may feel somewhat arbitrary; but after having introduced an order on $\mathcal{R}$, we will see that λ describes orders of magnitude, the relation $\approx$ corresponds to agreement up to infinitely small relative error, while $\sim$ corresponds to agreement of order of magnitude.

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The sets $\mathcal{R}$ and $\mathcal{C}$ are endowed with formal power series multiplication and componentwise addition, which make them into fields [14, 19] in which we can isomorphically embed $\mathbb{R}$ and $\mathbb{C}$ (respectively) as subfields via the map $E:\mathbb{R},\mathbb{C}\to\mathcal{R},\mathcal{C}$ defined by

$$E(x)[q]=\begin{cases} x & \text{if } q=0\\ 0 & \text{else}\end{cases}. \tag{1.1}$$

Definition 1.2. (Order on $\mathcal{R}$) Let $x,y\in\mathcal{R}$ be given. Then we say that $x>y$ (or $y<x$) if $x\neq y$ and $(x-y)[\lambda(x-y)]>0$; and we say $x\geq y$ (or $y\leq x$) if $x=y$ or $x>y$.

It follows that the relation $\geq$ (or $\leq$) defines a total order on $\mathcal{R}$ which makes it into an ordered field. Moreover, the embedding E in Equation (1.1) of $\mathbb{R}$ into $\mathcal{R}$ is compatible with the order.

The order leads to the definition of an ordinary absolute value on $\mathcal{R}$:

$$|x|_o=\max\{x,-x\}=\begin{cases} x & \text{if } x\geq 0\\ -x & \text{if } x<0;\end{cases}$$

which induces the same topology on $\mathcal{R}$ (called the order topology or valuation topology, and denoted by τ_v) as that induced by the ultrametric absolute value:

$$|x|=\begin{cases} e^{-\lambda(x)} & \text{if } x\neq 0\\ 0 & \text{if } x=0,\end{cases}$$

as was shown in [18]. Moreover, two corresponding absolute values are defined on $\mathcal{C}$ in the natural way: for $z=x+iy\in\mathcal{C}$, with $x,y\in\mathcal{R}$,

$$\begin{aligned} |z|_o &= \sqrt{x^2+y^2};\ \text{and}\\ |z| &= \begin{cases} e^{-\lambda(z)} & \text{if } z\neq 0\\ 0 & \text{if } z=0\end{cases} = \max\{|x|,|y|\}.\end{aligned}$$

Thus, $\mathcal{C}$ is topologically isomorphic to $\mathcal{R}^2$ provided with the product topology induced by $|\cdot|_o$ (or $|\cdot|$) in $\mathcal{R}$.

We note in passing here that $|\cdot|$ is a non-Archimedean valuation on $\mathcal{R}$ (resp. $\mathcal{C}$); that is, it satisfies the following properties

(1) $|v|\geq 0$ for all $v\in\mathcal{R}$ (resp. $v\in\mathcal{C}$) and $|v|=0$ if and only if $v=0$;
(2) $|vw|=|v||w|$ for all $v,w\in\mathcal{R}$ (resp. $v,w\in\mathcal{C}$); and
(3) $|v+w|\leq\max\{|v|,|w|\}$ for all $v,w\in\mathcal{R}$ (resp. $v,w\in\mathcal{C}$): the strong triangle inequality.

Thus, $(\mathcal{R},|\cdot|)$ and $(\mathcal{C},|\cdot|)$ are non-Archimedean valued fields. Moreover, the map $\Lambda:\mathcal{R}\times\mathcal{R}\to\mathbb{R}$ (resp. $\Lambda:\mathcal{C}\times\mathcal{C}\to\mathbb{R}$), given by

$$\Lambda(u,v)=\begin{cases} e^{-\lambda(u-v)} & \text{if } u\neq v\\ 0 & \text{if } u=v,\end{cases}$$

is an ultrametric on $\mathcal{R}$ (resp. $\mathcal{C}$) which makes it into an ultrametric space.

Besides the usual order relations on $\mathcal{R}$, some other notations are convenient.

Definition 1.3. ($\ll,\gg$) Let $x,y\in\mathcal{R}$ be non-negative. We say x is infinitely smaller than y (and write $x\ll y$) if $nx<y$ for all $n\in\mathbb{N}$; we say x is infinitely

larger than y (and write $x \gg y$) if $y \ll x$. If $x \ll 1$, we say x is infinitely small; if $x \gg 1$, we say x is infinitely large. Infinitely small numbers are also called infinitesimals or differentials. Infinitely large numbers are also called infinite. Non-negative numbers that are neither infinitely small nor infinitely large are also called finite.

Remark 1.4. For $\xi, \zeta \in \mathcal{R}$ (resp. $\xi, \zeta \in \mathcal{C}$), we have that

$$|\xi|_o \ll |\zeta|_o \Leftrightarrow |\xi| < |\zeta| \Leftrightarrow \lambda(\xi) > \lambda(\zeta).$$

Moreover, for $\xi \neq 0$ in $\mathbb{R}$ (resp. $\mathbb{C}$), we have that

$$\xi \sim |\xi|_o \sim 1 \text{ and } |\xi| = 1.$$

Definition 1.5. (The Number d) Let d be the element of $\mathcal{R}$ given by $d[1] = 1$ and $d[t] = 0$ for $t \neq 1$.

It follows that, given a rational number q, then d^q is given by

$$d^q[t] = \begin{cases} 1 & \text{if } t = q \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to check that $0 < d^q \ll 1$ (resp. $|d^q| < 1$) if $q > 0$, and $d^q \gg 1$ (resp. $|d^q| > 1$) if $q < 0$ in $\mathbb{Q}$. Moreover, for all $\xi \in \mathcal{R}$ (resp. $\mathcal{C}$), the elements of $\text{supp}(\xi)$ can be arranged in ascending order, say $\text{supp}(\xi) = \{q_1, q_2, \ldots\}$ with $q_j < q_{j+1}$ for all j; and ξ can be written as $\xi = \sum_{j=1}^{\infty} \xi[q_j] d^{q_j}$, where the series converges in the valuation topology.

Altogether, it follows that $\mathcal{R}$ (resp. $\mathcal{C}$) is a non-Archimedean field extension of $\mathbb{R}$ (resp. $\mathbb{C}$). For a detailed study of these fields, we refer the reader to the survey paper [19] and the references therein. In particular, it is shown that $\mathcal{R}$ and $\mathcal{C}$ are complete with respect to the natural (valuation) topology.

It follows therefore that the fields $\mathcal{R}$ and $\mathcal{C}$ are just special cases of the class of fields discussed in [13]. For a general overview of the algebraic properties of formal power series fields in general, we refer the reader to the comprehensive overview by Ribenboim [10], and for an overview of the related valuation theory to the books by Krull [8], Schikhof [13] and Alling [1]. A thorough and complete treatment of ordered structures can also be found in [9]. A more comprehensive survey of all non-Archimedean fields can be found in [3].

Besides being the smallest ordered non-Archimedean field extension of the real numbers that is both Cauchy complete in the order topology and real closed [14], the Levi-Civita field $\mathcal{R}$ is of particular interest because of its practical usefulness. Since the supports of the elements of $\mathcal{R}$ are left-finite, it is possible to represent these numbers on a computer; and having infinitely small numbers in the field allows for many computational applications [14, 7]. One such application is the computation of derivatives of real functions representable on a computer [16], where both the accuracy of formula manipulators and the speed of classical numerical methods are achieved. Similarly, $\mathcal{C}$ is the smallest non-Archimedean valued field extension of $\mathbb{C}$ that is Cauchy complete in the valuation topology and algebraically closed.

2. The Topological Structure of $\mathcal{C}$

In this section, we study two topologies on $\mathcal{C}$: one induced naturally by the valuation $|\cdot|$ mentioned in the introduction above, which we call the valuation topology, and another weaker topology induced by a family of seminorms, which we call weak topology.

2.1. **Valuation Topology τ_v.**

We start this subsection by recalling that the valuation topology is induced by the non-Archimedean valuation $|\cdot|:\mathcal{C}\to\mathbb{R}$ given by

$$|z| = \begin{cases} e^{-\lambda(z)} & \text{if } z\neq 0\\ 0 & \text{if } z=0\end{cases}$$

or, equivalently, by the ultrametric $\Lambda:\mathcal{C}\times\mathcal{C}\to\mathbb{R}$ given by $\Lambda(z,\xi)=|z-\xi|$.

Definition 2.1. For $z\in\mathcal{C}$, $r>0$ in $\mathbb{R}$, and $t>0$ in $\mathcal{R}$, let

$$\begin{aligned} B_v(z,r) &= \{\xi\in\mathcal{C}:|\xi-z|<r\}\\ B_v[z,r] &= \{\xi\in\mathcal{C}:|\xi-z|\le r\}\\ B_o(z,t) &= \{\xi\in\mathcal{C}:|\xi-z|_o<t\}\\ B_o[z,t] &= \{\xi\in\mathcal{C}:|\xi-z|_o\le t\}.\end{aligned}$$

It is easy to check that the family of sets

$$\tau_v := \{O\subset\mathcal{C}:\forall z\in O, \exists r>0 \text{ in } \mathbb{R} \text{ such that } B_v(z,r)\subset O\}$$

is indeed a topology on $\mathcal{C}$. Moreover,

$$\tau_v = \{A\subset\mathcal{C}:\forall z\in A, \exists t>0 \text{ in } \mathcal{R} \text{ such that } B_o(z,t)\subset A\}.$$

That is, the ordinary absolute value $|\cdot|_o$ and the non-Archimedean absolute value $|\cdot|$ induce the same topology, namely τ_v, on $\mathcal{C}$.

Definition 2.2. Let $A\subset\mathcal{C}$. Then we say that A is open in $(\mathcal{C},\tau_v)$ if $A\in\tau_v$. We say that A is closed in $(\mathcal{C},\tau_v)$ if its complement $\mathcal{C}\setminus A\in\tau_v$.

Like in any ultrametric space, each ball of the form $B_v(z_0,r)$ or $B_v[z_0,r]$ with $z_0\in\mathcal{C}$ and $r>0$ in $\mathbb{R}$, is both open and closed (clopen) in $(\mathcal{C},\tau_v)$ [3, Theorem 1.6].

Definition 2.3. Let $A\subset\mathcal{C}$. Then we say that A is compact in $(\mathcal{C},\tau_v)$ if every open cover of A in $(\mathcal{C},\tau_v)$ has a finite subcover.

Remark 2.4. Since τ_v is induced by a metric on $\mathcal{C}$, it follows by the Borel-Lebesgue Theorem (see for example [5, Section 9.2]) that A is compact in $(\mathcal{C},\tau_v)$ if and only if A is sequentially compact.

Theorem 2.5. *$(\mathcal{C},\tau_v)$ is a totally disconnected topological space. It is Hausdorff and not locally compact. There are no countable bases. The topology induced to $\mathbb{C}$ is the discrete topology.*

Proof. Let $A\subset\mathcal{C}$ contain more than one point; and let $\zeta\neq\xi$ in A be given. Let

$$G_1=\{z\in\mathcal{C}:|z-\xi|<|\zeta-\xi|\} \text{ and } G_2=\mathcal{C}\setminus G_1.$$

Then G_1 and G_2 are disjoint and open in $(\mathcal{C}, \tau_v)$; $\xi \in G_1 \cap A$ and $\zeta \in G_2 \cap A$; and $A \subset G_1 \cup G_2 = \mathcal{C}$. This shows that any subset of $(\mathcal{C}, \tau_v)$ containing more than one point is disconnected; and hence $(\mathcal{C}, \tau_v)$ is totally disconnected. It follows that $(\mathcal{C}, \tau_v)$ is Hausdorff. That $(\mathcal{C}, \tau_v)$ is Hausdorff also follows from the fact that it is a metric space [6, p. 66, Problem 7(a)].

To prove that $(\mathcal{C}, \tau_v)$ is not locally compact, let $z \in \mathcal{C}$ be given and let U be a neighborhood of z. We show that the closure $\bar{U}$ of U is not compact. Let $\epsilon > 0$ in $\mathbb{R}$ be such that $\ln \epsilon \in \mathbb{Q}$ and $B_v(z, \epsilon) \subset U$; and consider the sets

$$\begin{aligned} M_0 &= \mathcal{C} \setminus B_v(z, \epsilon); \\ M_n &= \{\xi \in \mathcal{C} : -\ln \epsilon + n - 1 < \lambda(\xi - z) \leq -\ln \epsilon + n\} \text{ for } n \in \mathbb{N}. \end{aligned}$$

Then it is easy to check that M_n is open in $(\mathcal{C}, \tau_v)$ for all $n \geq 0$ and that $\bigcup_{n=1}^{\infty} M_n = \{\xi \in \mathcal{C} : \lambda(\xi - z) > -\ln \epsilon\} = B_v(z, \epsilon)$. It follows that $\bigcup_{n=0}^{\infty} M_n = \mathcal{C}$ and hence $\bar{U} \subset \bigcup_{n=0}^{\infty} M_n$. But it is impossible to select finitely many of the M_n's to cover $\bar{U}$ because each of the infinitely many elements $\xi_n := z + d^{-\ln \epsilon + n}$ of $\bar{U}$, $n = 1, 2, 3, \ldots$, is contained only in the set M_n.

There can not be any countable bases because the uncountably many open sets $M_Z = B_v(Z, 1/2)$, with $Z \in \mathbb{C}$, are disjoint. The open sets induced on $\mathbb{C}$ by the sets M_Z are just the singletons $\{Z\}$. Thus, in the induced topology, all sets are open and the topology is therefore discrete. □

Remark 2.6. A detailed study of the properties in Theorem 2.5 reveals that they hold in an identical way in any non-Archimedean valued field, and thus the above unusual properties are not specific to $\mathcal{C}$.

As an immediate consequence of the fact that $(\mathcal{C}, \tau_v)$ is not locally compact, we obtain the following result.

Corollary 2.7. *None of the balls $B_v(z_0, r)$, $B_v[z_0, r]$, $B_o(z_0, t)$, or $B_o[z_0, t]$ are compact in $(\mathcal{C}, \tau_v)$ for all $z_0 \in \mathcal{C}$, $r > 0$ in $\mathbb{R}$ and $t > 0$ in $\mathcal{R}$.*

Since τ_v is induced on $\mathcal{C}$ by the ultrametric valuation $|\cdot|$, we define boundedness of a set in $(\mathcal{C}, \tau_v)$ as follows.

Definition 2.8. Let $A \subset \mathcal{C}$. Then we say that A is bounded in $(\mathcal{C}, \tau_v)$ if there exists $M > 0$ in $\mathbb{R}$ such that $|z| \leq M$ for all $z \in A$.

Proposition 2.9. *Let A be compact in $(\mathcal{C}, \tau_v)$. Then A is closed and bounded in $(\mathcal{C}, \tau_v)$. Moreover, A has an empty interior in $(\mathcal{C}, \tau_v)$; that is,*

$$int_v(A) := \{a \in A : \exists r > 0 \text{ in } \mathbb{R} \ni B_v(a, r) \subset A\} = \emptyset.$$

Proof. That A is closed in $(\mathcal{C}, \tau_v)$ follows from the fact that $(\mathcal{C}, \tau_v)$ is a Hausdorff topological space and A is compact in $(\mathcal{C}, \tau_v)$ [11, p. 36].

Now we show that A is bounded in $(\mathcal{C}, \tau_v)$. For each $n \in \mathbb{N}$, let $G_n = B_v(0, n)$. Then, for each $n \in \mathbb{N}$, G_n is open in $(\mathcal{C}, \tau_v)$. Moreover, $A \subset \bigcup_{n \in \mathbb{N}} G_n = \mathcal{C}$. Since A is compact in $(\mathcal{C}, \tau_v)$, we can choose a finite subcover; thus, there is $m \in \mathbb{N}$ and there exist $j_1 < j_2 < \ldots < j_m$ in $\mathbb{N}$ such that

$$A \subset \bigcup_{l=1}^{m} G_{j_l} = G_{j_m} = B_v(0, j_m).$$

It follows that $|z| < j_m$ for all $z \in A$, and hence A is bounded in $(\mathcal{C}, \tau_v)$.

Finally, we show that $\mathrm{int}_v(A) = \emptyset$. Assume to the contrary that $\mathrm{int}_v(A) \neq \emptyset$. Then there exist $z_0 \in A$ and $r > 0$ in $\mathbb{R}$ such that $B_v(z_0, r) \subset A$. Since $B_v(z_0, r)$ is a closed subset of the compact set A, it follows that $B_v(z_0, r)$ is compact in $(\mathcal{C}, \tau_v)$, which contradicts Corollary 2.7. □

The following examples show that there are countably infinite closed and bounded sets that are not compact and there are uncountable sets that are compact in $(\mathcal{C}, \tau_v)$.

Example 2.10. Let $A = [0, 1] \cap \mathbb{Q}$. Then, clearly, A is countably infinite and bounded in $(\mathcal{C}, \tau_v)$. We show that A is closed in $(\mathcal{C}, \tau_v)$. Let $z \in \mathcal{C} \setminus A$ be given and let $G_0 = B_v(z, 1/2)$. If $G_0 \cap A \neq \emptyset$ then there exists $q \in A$ such that $G_0 \cap A = \{q\}$. Let $r = |q - z|$ and let $G = B_v(z, r)$. Then G is open in $(\mathcal{C}, \tau_v)$ and $G \cap A = \emptyset$. Thus, $\mathcal{C} \setminus A$ is open, and hence A is closed in $(\mathcal{C}, \tau_v)$.

Next we show that A is not compact in $(\mathcal{C}, \tau_v)$. For each $q \in A$, let $G_q = B_v(q, 1/2)$. Then G_q is open in $(\mathcal{C}, \tau_v)$ for each q and $A \subset \bigcup_{q \in A} G_q$, but we can't select a finite subcover since each $t \in A$ is contained only in G_t.

Example 2.11. Let $C_{\mathcal{R}}$ denote the Cantor-like set constructed in the same way as the standard real Cantor set C; but instead of deleting the middle third, we delete from the middle an open interval $(1 - 2d)$ times the size of each of the closed subintervals of $[0, 1]$ at each step of the construction. Then $C_{\mathcal{R}}$ is compact in $(\mathcal{C}, \tau_v)$.

It turns out that if we view $\mathcal{C}$ as an infinite dimensional vector space over $\mathbb{C}$ then τ_v is not a vector topology; that is, $(\mathcal{C}, \tau_v)$ is not a linear topological space.

Theorem 2.12. *τ_v is not a vector topology.*

Proof. Assume to the contrary that τ_v is a vector topology. Then, by continuity of scalar multiplication, there exists an open set $O_{\mathbb{C}} \subset \mathbb{C}$ and there exists an open set $O_{\mathcal{C}} \subset \mathcal{C}$ such that $\alpha z \in B_v(1, 1/2)$ for all $\alpha \in O_{\mathbb{C}}$ and for all $z \in O_{\mathcal{C}}$. Let $\alpha_0 \in O_{\mathbb{C}}$ and $z_0 \in O_{\mathcal{C}}$ be given. Since $O_{\mathbb{C}}$ is open in $\mathbb{C}$, there exists $r > 0$ in $\mathbb{R}$ such that $B_{\mathbb{C}}(\alpha_0, 2r) := \{\beta \in \mathbb{C} : |\beta - \alpha_0|_o < 2r\} \subset O_{\mathbb{C}}$. Hence

$$\alpha_0 z_0 \in B_v(1, 1/2) \text{ and } (\alpha_0 + r) z_0 \in B_v(1, 1/2).$$

Since $\alpha_0 z_0 \in B_v(1, 1/2)$, it follows that $|\alpha_0 z_0 - 1| < \frac{1}{2}$ and hence $|z_0| = |\alpha_0 z_0| = 1$. Using the strong triangle inequality, we obtain that

$$\begin{aligned} |r z_0| &= |[(\alpha_0 + r) z_0 - 1] - [\alpha_0 z_0 - 1]| \\ &\leq \max\{|(\alpha_0 + r) z_0 - 1|, |\alpha_0 z_0 - 1|\} < \frac{1}{2}, \end{aligned}$$

which contradicts the fact that $|r z_0| = 1$, since $|r| = 1 = |z_0|$. □

Since any normed vector space, with the metric topology induced by its norm, is a linear topological space ([4] Proposition III.1.3), we readily infer from Theorem 2.12 that there can be no norm on $\mathcal{C}$ that would induce the same topology as τ_v on $\mathcal{C}$.

2.2. **Weak Topology.** In the following, we will think of $\mathcal{C}$ as an infinite-dimensional vector space over $\mathbb{C}$. We define a family of semi-norms on $\mathcal{C}$, which induces a topology weaker than the valuation topology, called the weak topology.

Definition 2.13. Given $r \in \mathbb{R}$, we define a mapping $\|\cdot\|_r : \mathcal{C} \to \mathbb{R}$ as follows: $\|z\|_r = \max\{|z[q]|_o : q \in \mathbb{Q} \text{ and } q \leq r\}$.

The maximum in Definition 2.13 exists in $\mathbb{R}$ since, for any $r \in \mathbb{R}$, only finitely many of the $z[q]$'s considered do not vanish.

Definition 2.14. For $z \in \mathcal{C}$ and $r > 0$ in $\mathbb{R}$, we define

$$\begin{aligned} B_w(z,r) &= \{\xi \in \mathcal{C} : \|\xi - z\|_{1/r} < r\} \text{ and} \\ B_w[z,r] &= \{\xi \in \mathcal{C} : \|\xi - z\|_{1/r} \leq r\}. \end{aligned}$$

Lemma 2.15. *Let $0 < r_2 < r_1$ be given in $\mathbb{R}$, let $r = \min\{r_2, r_1 - r_2\}$, and let $z \in \mathcal{C}$ be given. Then for all $\xi \in B_w(z,r)$, we have that $B_w(\xi, r_2) \subset B_w(z, r_1)$. In particular, $B_w(z, r_2) \subset B_w(z, r_1)$.*

Proof. Let $\xi \in B_w(z,r)$ be given; we show that $B_w(\xi, r_2) \subset B_w(z, r_1)$. So let $\zeta \in B_w(\xi, r_2)$ be given. Then $\|\zeta - \xi\|_{1/r_2} < r_2$. It follows that

$$\begin{aligned} \|\zeta - z\|_{1/r_1} &\leq \|\zeta - z\|_{1/r_2} \leq \|\zeta - \xi\|_{1/r_2} + \|\xi - z\|_{1/r_2} \\ &< r_2 + \|\xi - z\|_{1/r_2} \\ &\leq r_2 + \|\xi - z\|_{1/r} \\ &< r_2 + r \leq r_2 + (r_1 - r_2) \\ &= r_1. \end{aligned}$$

Thus $\zeta \in B_w(z, r_1)$ for all $\zeta \in B_w(\xi, r_2)$; and hence $B_w(\xi, r_2) \subset B_w(z, r_1)$.

Finally, since $z \in B_w(z,r)$, it follows that $B_w(z, r_2) \subset B_w(z, r_1)$. ☐

Proposition 2.16. *The family of subsets of $\mathcal{C}$*

$$\tau_w := \{O \subset \mathcal{C} : \forall z \in O, \exists r > 0 \text{ in } \mathbb{R} \text{ such that } B_w(z,r) \subset O\}$$

is a topology on $\mathcal{C}$.

Proof. Let $\{O_\alpha\}_{\alpha \in A}$ be a collection of elements of τ_w. We show that $\bigcup_{\alpha \in A} O_\alpha \in \tau_w$. So let $z \in \bigcup_{\alpha \in A} O_\alpha$ be given. Then there exists $\alpha_0 \in A$ such that $z \in O_{\alpha_0}$. Since $O_{\alpha_0} \in \tau_w$, there exists $r > 0$ in $\mathbb{R}$ such that $B_w(z,r) \subset O_{\alpha_0}$. Thus, $B_w(z,r) \subset \bigcup_{\alpha \in A} O_\alpha$.

Next we show that τ_w is closed under finite intersections: It suffices to show that if $O_1, O_2 \in \tau_w$ then $O_1 \cap O_2 \in \tau_w$. So let $O_1, O_2 \in \tau_w$ and let $z \in O_1 \cap O_2$ be given. Then there exist $r_1, r_2 > 0$ in $\mathbb{R}$ such that $B_w(z, r_1) \subset O_1$ and $B_w(z, r_2) \subset O_2$. Let $r = \min\{r_1, r_2\}$. Then, using Lemma 2.15, we obtain that $B_w(z,r) \subset B_w(z, r_1) \subset O_1$ and $B_w(z,r) \subset B_w(z, r_2) \subset O_2$. Thus, $B_w(z,r) \subset O_1 \cap O_2$.

That $\emptyset$ and $\mathcal{C}$ are both elements of τ_w is clear. It follows that τ_w is a topology on $\mathcal{C}$ and hence $(\mathcal{C}, \tau_w)$ is a topological space. ☐

As Theorem 2.17 and Theorem 2.18 below will show, there is a translation invariant metric on $\mathcal{C}$ that induces the topology τ_w on $\mathcal{C}$.

Theorem 2.17. *The map* $\Delta : \mathcal{C} \times \mathcal{C} \to \mathbb{R}$, *given by*

$$\Delta(z,\xi) = \sum_{k=1}^{\infty} 2^{-k} \frac{\|z-\xi\|_k}{1+\|z-\xi\|_k}, \tag{2.1}$$

is a translation invariant metric.

Proof. Δ is positive-definite: It is clear that $\Delta(z,\xi) \geq 0$ for all $z, \xi \in \mathcal{C}$. Moreover, for all $z, \xi \in \mathcal{C}$,

$$\begin{aligned} \Delta(z,\xi) = 0 \quad &\Leftrightarrow \quad \|z-\xi\|_k = 0 \text{ for all } k \in \mathbb{N} \\ &\Leftrightarrow \quad (z-\xi)[q] = 0 \text{ for all } q \leq k \text{ in } \mathbb{Q}, \text{ for all } k \in \mathbb{N} \\ &\Leftrightarrow \quad (z-\xi)[q] = 0 \text{ for all } q \in \mathbb{Q} \\ &\Leftrightarrow \quad z = \xi. \end{aligned}$$

Δ is symmetric: For all $z, \xi \in \mathcal{C}$, we have that

$$\Delta(z,\xi) = \sum_{k=1}^{\infty} 2^{-k} \frac{\|z-\xi\|_k}{1+\|z-\xi\|_k} = \sum_{k=1}^{\infty} 2^{-k} \frac{\|\xi-z\|_k}{1+\|\xi-z\|_k} = \Delta(\xi,z).$$

Δ satisfies the triangle inequality: Let $\xi, \zeta, z \in \mathcal{C}$ be given. Then, for all $k \in \mathbb{N}$, we have:

$$\begin{aligned} \frac{\|\xi-\zeta\|_k}{1+\|\xi-\zeta\|_k} &= 1 - \frac{1}{1+\|\xi-\zeta\|_k} \leq 1 - \frac{1}{1+\|\xi-z\|_k+\|\zeta-z\|_k} \\ &= \frac{\|\xi-z\|_k}{1+\|\xi-z\|_k+\|\zeta-z\|_k} + \frac{\|\zeta-z\|_k}{1+\|\xi-z\|_k+\|\zeta-z\|_k} \\ &\leq \frac{\|\xi-z\|_k}{1+\|\xi-z\|_k} + \frac{\|\zeta-z\|_k}{1+\|\zeta-z\|_k}. \end{aligned}$$

Thus,

$$\begin{aligned} \Delta(\xi,\zeta) &= \sum_{k=1}^{\infty} 2^{-k} \frac{\|\xi-\zeta\|_k}{1+\|\xi-\zeta\|_k} \\ &\leq \sum_{k=1}^{\infty} 2^{-k} \frac{\|\xi-z\|_k}{1+\|\xi-z\|_k} + \sum_{k=1}^{\infty} 2^{-k} \frac{\|\zeta-z\|_k}{1+\|\zeta-z\|_k} \\ &= \Delta(\xi,z) + \Delta(\zeta,z). \end{aligned}$$

Finally, for all $\xi, \zeta, z \in \mathcal{C}$, we have:

$$\begin{aligned} \Delta(\xi+z,\zeta+z) &= \sum_{k=1}^{\infty} 2^{-k} \frac{\|(\xi+z)-(\zeta+z)\|_k}{1+\|(\xi+z)-(\zeta+z)\|_k} \\ &= \sum_{k=1}^{\infty} 2^{-k} \frac{\|\xi-\zeta\|_k}{1+\|\xi-\zeta\|_k} \\ &= \Delta(\xi,\zeta). \end{aligned}$$

□

Next we will show that the metric Δ introduced above induces the same topology on $\mathcal{C}$ as the weak topology τ_w.

Theorem 2.18. *Let τ_Δ denote the topology induced by the metric Δ in Equation (2.1). Then $\tau_\Delta = \tau_w$.*

Proof. First we show that $\tau_\Delta \subseteq \tau_w$: Let $O \in \tau_\Delta$, and let $z \in O$ be given. Then there exists $r > 0$ in $\mathbb{R}$ such that

$$B_\Delta(z, r) := \{\xi \in \mathcal{C} : \Delta(z, \xi) < r\} \subset O.$$

Let $j \in \mathbb{N}$ be such that $j > 2/r$. Then

$$2^{-j} < \frac{1}{j} < \frac{r}{2}.$$

We show that $B_w(z, 1/j) \subset O$: Let $\xi \in B_w(z, 1/j)$ be given. Then $\|z - \xi\|_j < 1/j$. It follows that

$$\|z - \xi\|_k < \frac{1}{j} \leq \frac{1}{k} \text{ for } 1 \leq k \leq j.$$

Thus,

$$\begin{aligned}
\Delta(z, \xi) &= \sum_{k=1}^{\infty} 2^{-k} \frac{\|z - \xi\|_k}{1 + \|z - \xi\|_k} \\
&= \sum_{k=1}^{j} 2^{-k} \frac{\|z - \xi\|_k}{1 + \|z - \xi\|_k} + \sum_{k=j+1}^{\infty} 2^{-k} \frac{\|z - \xi\|_k}{1 + \|z - \xi\|_k} \\
&\leq \sum_{k=1}^{j} 2^{-k} \|z - \xi\|_k + \sum_{k=j+1}^{\infty} 2^{-k} \\
&< \frac{1}{j} \sum_{k=1}^{j} 2^{-k} + 2^{-j} \sum_{k=1}^{\infty} 2^{-k} \\
&< \frac{1}{j} + 2^{-j} \\
&< \frac{r}{2} + \frac{r}{2} = r.
\end{aligned}$$

Hence $\xi \in B_\Delta(z, r) \subset O$. Thus, $B_w(z, 1/j) \subset O$. This shows that $O \in \tau_w$.

Next we show that $\tau_w \subseteq \tau_\Delta$: Let $O \in \tau_w$; and let $z \in O$ be given. Then there exists $M \in \mathbb{R}$ such that $0 < M < 1$ and $B_w(z, M) \subset O$. Choose $j \in \mathbb{N}$ such that $j > 1/M$. We show that $B_\Delta\left(z, M2^{-(j+1)}\right) \subset O$. So let $\xi \in B_\Delta\left(z, M2^{-(j+1)}\right)$ be given. Then

$$\Delta(z, \xi) = \sum_{k=1}^{\infty} 2^{-k} \frac{\|z - \xi\|_k}{1 + \|z - \xi\|_k} < M2^{-(j+1)}.$$

Thus,

$$2^{-j} \frac{\|z - \xi\|_j}{1 + \|z - y\|_j} < \frac{M}{2} 2^{-j}; \text{ and hence } \frac{\|z - \xi\|_j}{1 + \|z - \xi\|_j} < \frac{M}{2}.$$

It follows that
$$\|z-\xi\|_j < \frac{M}{2-M} < M \text{ since } 0<M<1.$$
Therefore,
$$\|z-\xi\|_{1/M} \le \|z-\xi\|_j < M;$$
and hence $\xi \in B_w(z,M) \subset O$. Thus, $B_\Delta\left(z, M2^{-(j+1)}\right) \subset O$. This shows that $O \in \tau_\Delta$. □

It turns out that the weak topology is the most useful topology for considering convergence of sequences and series in general; see [15] and the references therein. Moreover, it is of great importance for the implementation of the $\mathcal{R}$ calculus on computers [16].

Definition 2.19. Let $A \subset \mathcal{C}$. Then we say that A is open in $(\mathcal{C}, \tau_w)$ if $A \in \tau_w$. We say that A is closed in $(\mathcal{C}, \tau_w)$ if its complement $\mathcal{C} \setminus A \in \tau_w$.

Since, by Theorem 2.18, τ_w is induced by a metric on $\mathcal{C}$ we define compactness in $(\mathcal{C}, \tau_w)$ just as we did in $(\mathcal{C}, \tau_v)$- see Definition 2.3- and as in any other metric space. Moreover, the following result follows readily.

Proposition 2.20. *Let $A \subset \mathcal{C}$. Then A is closed in $(\mathcal{C}, \tau_w)$ if and only if whenever $(a_n)_{n\in\mathbb{N}}$ is a sequence of elements in A that converges in $(\mathcal{C}, \tau_w)$ to $a \in \mathcal{C}$, then $a \in A$.*

Proposition 2.21. *$(\mathcal{C}, \tau_w)$ is a Hausdorff topological space. The topology induced on $\mathbb{C}$ by the weak topology is the usual topology on $\mathbb{C}$.*

Proof. That $(\mathcal{C}, \tau_w)$ is a Hausdorff topological space follows from the fact that it is a metric space.

Considering elements of $\mathbb{C}$, their supports (when viewed as elements of $\mathcal{C}$) are all equal to $\{0\}$. Therefore, the open sets in $(\mathcal{C}, \tau_w)$ correspond to the open subsets of $\mathbb{C}$ in its usual topology. □

Proposition 2.22. *Let $G \subset \mathcal{C}$ be open in $(\mathcal{C}, \tau_w)$. Then G is open in $(\mathcal{C}, \tau_v)$.*

Proof. Let $z \in G$ be given. Then there exists $r>0$ in $\mathbb{R}$ such that $B_w(z,r) \subset G$. Let $n \in \mathbb{N}$ be such that $n > 1/r$. We show that $B_v\left(z, e^{-n}\right) \subset G$.

Let $\xi \in B_v\left(z, e^{-n}\right)$ be given. Then $|\xi - z| < e^{-n}$. Thus, $e^{-\lambda(\xi - z)} < e^{-n}$ and hence $\lambda(\xi - z) > n$. It follows that $(\xi - z)[q] = 0$ for all $q < n$. In particular, $(\xi - z)[q] = 0$ for all $q \le 1/r$; and hence $\|\xi - z\|_{1/r} = 0 < r$. Thus, $\xi \in B_w(z,r) \subset G$ for all $\xi \in B_v\left(z, e^{-n}\right)$. It follows that $B_v\left(z, e^{-n}\right) \subset G$, and hence G is open in $(\mathcal{C}, \tau_v)$. □

The following example shows that the converse of Proposition 2.22 is not true.

Example 2.23. The ball $B_v(0,1)$ is open in $(\mathcal{C}, \tau_v)$; but we show that it is not open in $(\mathcal{C}, \tau_w)$. Let $r > 0$ in $\mathbb{R}$ be given. Let $z = (r/2)d^{-1}$; then $z \notin B_v(0,1)$ since $|z - 0| = |z| = e > 1$, but $z \in B_w(0,r)$ since $\|z\|_{1/r} = r/2 < r$. It follows that $B_w(0,r) \not\subset B_v(0,1)$ for all $r>0$; and hence $B_v(0,1)$ is not open in $(\mathcal{R}, \tau_w)$.

Remark 2.24. Similarly, we can show that none of the balls $B_v(z_0, r)$, $B_v[z_0, r]$, $B_o(z_0, t)$, or $B_o[z_0, t]$ are open in $(\mathcal{C}, \tau_w)$ for all $z_0 \in \mathcal{C}$, $r > 0$ in $\mathbb{R}$ and $t > 0$ in $\mathcal{R}$.

It follows from Proposition 2.22 and Example 2.23 that the weak topology is strictly weaker than the valuation topology ($\tau_w \subsetneq \tau_v$).

Corollary 2.25. *Let $A \subset \mathcal{C}$ be closed in $(\mathcal{C}, \tau_w)$. Then A is closed in $(\mathcal{C}, \tau_v)$.*

Corollary 2.26. *Let $A \subset \mathcal{C}$ be compact in $(\mathcal{C}, \tau_v)$. Then A is compact in $(\mathcal{C}, \tau_w)$.*

One of the advantages of the weak topology τ_w over the valuation topology τ_v is that the former is a vector topology as the following theorem shows while the latter is not (Theorem 2.12).

Theorem 2.27. *$(\mathcal{C}, \tau_w)$ is a linear topological space; that is, τ_w is a vector topology.*

Proof. First we show that $+$ is continuous on $(\mathcal{C}, \tau_w) \times (\mathcal{C}, \tau_w)$. Let O be open in $(\mathcal{C}, \tau_w)$. We need to show that the inverse image A of O under $+$ is open in $(\mathcal{C}, \tau_w) \times (\mathcal{C}, \tau_w)$. So let $(z_1, z_2) \in A$ be given. Then $z_1 + z_2 \in O$. Since O is open in $(\mathcal{C}, \tau_w)$, there exists $r > 0$ in $\mathbb{R}$ such that $B_w(z_1 + z_2, r) \subset O$. Now let $\xi \in B_w(z_1, r/2)$ and $\zeta \in B_w(z_2, r/2)$ be given. Then

$$\begin{aligned}
\|\xi + \zeta - (z_1 + z_2)\|_{1/r} &\leq \|\xi - z_1\|_{1/r} + \|\zeta - z_2\|_{1/r} \\
&\leq \|\xi - z_1\|_{2/r} + \|\zeta - z_2\|_{2/r} \\
&< \frac{r}{2} + \frac{r}{2} = r.
\end{aligned}$$

Thus, $\xi + \zeta \in B_w(z_1 + z_2, r) \subset O$; and hence $(\xi, \zeta) \in A$. It follows that $B_w(z_1, r/2) \times B_w(z_2, r/2) \subset A$. Hence A is open in $(\mathcal{C}, \tau_w) \times (\mathcal{C}, \tau_w)$.

Next we show that scalar multiplication $\cdot : \mathbb{C} \times (\mathcal{C}, \tau_w) \to (\mathcal{C}, \tau_w)$ is continuous. Let O be open in $(\mathcal{C}, \tau_w)$ and let S denote the inverse image of O under $\cdot$. We show that S is open in $\mathbb{C} \times (\mathcal{C}, \tau_w)$. So let $(\alpha, z) \in S$ be given. Then $\alpha z \in O$. Hence there exists $r > 0$ in $\mathbb{R}$ such that $B_w(\alpha z, r) \subset O$.

First assume that $\alpha = 0$, then $\alpha z = 0$. As a first subcase, assume that $\|z\|_{1/r} = 0$. Then we claim that $B_{\mathbb{C}}(0, 1) \times B_w(z, r) \subset S$: Let $\beta \in B_{\mathbb{C}}(0, 1)$ and $\xi \in B_w(z, r)$ be given. Then

$$\begin{aligned}
\|\beta\xi\|_{1/r} &= |\beta|_o \|\xi\|_{1/r} < \|\xi\|_{1/r} \\
&\leq \|\xi - z\|_{1/r} + \|z\|_{1/r} \\
&= \|\xi - z\|_{1/r} \\
&< r.
\end{aligned}$$

Thus, $\beta\xi \in B_w(0, r) \subset O$ and hence $(\beta, \xi) \in S$. As a second subcase, assume that $\|z\|_{1/r} \neq 0$. Let

$$r_1 = \min\left\{\frac{1}{2}, \frac{r}{2\|z\|_{1/r}}\right\}.$$

Then $r_1 > 0$ and $r_1 \in \mathbb{R}$. We claim that $B_{\mathbb{C}}(0, r_1) \times B_w(z, r) \subset S$: Let $\beta \in B_{\mathbb{C}}(0, r_1)$ and $\xi \in B_w(z, r)$ be given. Then

$$\begin{aligned} \|\beta\xi\|_{1/r} &= \|\beta(\xi - z) + \beta z\|_{1/r} \\ &\leq |\beta|_o\|\xi - z\|_{1/r} + |\beta|_o\|z\|_{1/r} \\ &< r_1 r + r_1\|z\|_{1/r} \\ &\leq \frac{1}{2}r + \frac{r}{2\|z\|_{1/r}}\|z\|_{1/r} = r. \end{aligned}$$

Thus, $\beta\xi \in B_w(0, r) \subset O$ and hence $(\beta, \xi) \in S$.

Now assume that $\alpha \neq 0$. Let

$$r_1 = \min\left\{\frac{r}{2}, \frac{r}{2|\alpha|_o}\right\}$$

and

$$\eta = \begin{cases} 1/2 & \text{if } \|z\|_{1/r} = 0 \\ \\ \min\left\{\frac{1}{2}, \frac{r}{4\|z\|_{1/r}}\right\} & \text{if } \|z\|_{1/r} \neq 0. \end{cases}$$

Then $r_1 > 0$ and $\eta > 0$ in $\mathbb{R}$. We claim that $B_{\mathbb{C}}(\alpha, \eta) \times B_w(z, r_1) \subset S$: Let $\beta \in B_{\mathbb{C}}(\alpha, \eta)$ and $\xi \in B_w(z, r_1)$ be given. Then

$$\begin{aligned} \|\beta\xi - \alpha z\|_{1/r} &= \|(\beta - \alpha)(\xi - z) + (\beta - \alpha)z + \alpha(\xi - z)\|_{1/r} \\ &\leq |\beta - \alpha|_o\|\xi - z\|_{1/r} + |\beta - \alpha|_o\|z\|_{1/r} + |\alpha|_o\|\xi - z\|_{1/r}. \end{aligned}$$

Since $r_1 \leq r/2 < r$, we have:

$$\|\xi - z\|_{1/r} \leq \|\xi - z\|_{1/r_1} < r_1 \leq \frac{r}{2|\alpha|_o}; \text{ and hence } |\alpha|_o\|\xi - z\|_{1/r} < \frac{r}{2}.$$

Also

$$|\beta - \alpha|_o\|\xi - z\|_{1/r} < |\beta - \alpha|_o r_1 < \eta\frac{r}{2} \leq \frac{r}{4};$$

and

$$|\beta - \alpha|_o\|z\|_{1/r} \leq \eta\|z\|_{1/r} \leq \frac{r}{4}.$$

Altogether, we get that

$$\|\beta\xi - \alpha z\|_{1/r} < \frac{r}{4} + \frac{r}{4} + \frac{r}{2} = r.$$

Thus, $\beta\xi \in B_w(\alpha z, r) \subset O$ and hence $(\beta, \xi) \in S$. □

Because of the continuity of addition, it is easy to see that the mapping of translation by a fixed $z_0 \in \mathcal{C}$ (that is, the map $z \mapsto z + z_0$, $z \in \mathcal{C}$) is a homeomorphism of $\mathcal{C}$ onto itself. For this reason, the neighborhood structure at any point of $\mathcal{C}$ is the same as the neighborhood structure at 0; and it is sufficient to study the neighborhoods of 0 (henceforth referred to as the zero-neighborhoods.) Before we start our discussion of the zero-neighborhoods, we recall the following definitions.

Definition 2.28. Let $A \subset \mathcal{C}$. Then

(a) We say that A is circled if $\alpha z \in A$ for every $z \in A$ and every $\alpha \in \mathbb{C}$ satisfying $|\alpha|_o \leq 1$.
(b) We say that A is absorbing if for every $z \in \mathcal{C}$ there exists $\delta > 0$ in $\mathbb{R}$ such that $tz \in A$ for every $t \in \mathbb{C}$ satisfying $|t|_o \leq \delta$.

Lemma 2.29. *For all $r > 0$ in $\mathbb{R}$, the ball $B_w(0, r) \subset \mathcal{C}$ is circled and absorbing.*

Proof. Let $r > 0$ in $\mathbb{R}$ be given. First we show that $B_w(0, r)$ is circled. So let $z \in B_w(0, r)$ and let $\alpha \in \mathbb{C}$ be such that $|\alpha|_o \leq 1$. Then

$$\|\alpha z\|_{1/r} = |\alpha|_o \|z\|_{1/r} \leq \|z\|_{1/r} < r;$$

and hence $\alpha z \in B_w(0, r)$.

Next we show that $B_w(0, r)$ is absorbing. So let $z \in \mathcal{C}$ be given. We need to find $\delta > 0$ in $\mathbb{R}$ such that $tz \in B_w(0, r)$ for every $t \in \mathbb{C}$ satisfying $|t|_o \leq \delta$. Let

$$\delta = \begin{cases} \frac{r}{2\|z\|_{1/r}} & \text{if } \|z\|_{1/r} \neq 0 \\ \\ 1 & \text{if } \|z\|_{1/r} = 0. \end{cases}$$

Then $\delta > 0$ in $\mathbb{R}$. Moreover, for $t \in \mathbb{C}$ satisfying $|t|_o \leq \delta$, we have:

$$\|tz\|_{1/r} = |t|_o \|z\|_{1/r} \leq \delta \|z\|_{1/r} < r;$$

and hence $tz \in B_w(0, r)$. □

Of the family of circled and absorbing open balls $\{B_w(0, r) : 0 < r \in \mathbb{R}\}$, we can select a countable local base for the topology τ_w at 0.

Proposition 2.30. *$\{B_w(0, q) : 0 < q \in \mathbb{Q}\}$ is a local base for τ_w at 0.*

Proof. We need to show that for each $O \in \tau_w$ that contains 0, there exists $q > 0$ in $\mathbb{Q}$ such that $B_w(0, q) \subset O$. So let $O \in \tau_w$ be given such that $0 \in O$. Then there exists $r > 0$ in $\mathbb{R}$ such that $B_w(0, r) \subset O$. Let $q \in \mathbb{Q}$ be such that $0 < q < r$. Then it follows from Lemma 2.15 that $B_w(0, q) \subset B_w(0, r) \subset O$. □

Corollary 2.31. *$\{B_w(0, q) : 0 < q \in \mathbb{Q}\}$ is a countable base for the zero-neighborhoods in $(\mathcal{C}, \tau_w)$. That is, for each zero-neighborhood N there exists $q > 0$ in $\mathbb{Q}$ such that $B_w(0, q) \subset N$.*

Remark 2.32. It follows from the above discussion of the open weak balls $B_w(0, r)$ that $B_w[0, r]$ too is a circled and absorbing zero-neighborhood for each $r > 0$ in $\mathbb{R}$. Moreover, $\{B_w[0, q] : 0 < q \in \mathbb{Q}\}$ is a countable base for the zero-neighborhoods in $\mathcal{C}$.

Recall that in a Banach space a set is called bounded if it is bounded in norm. However, the appropriate generalization of this is not so obvious for spaces with no norm. Even in metric spaces problems can arise. If we try to mimic the Banach space situation and say that a set is bounded in $(\mathcal{C}, \tau_w)$ if and only if it is contained in some metric ball (using for example the metric of Theorem 2.17 which, by Theorem 2.18, induces the topology τ_w on $\mathcal{C}$), then we have a problem: $\mathcal{C}$ and hence any subset of $\mathcal{C}$ is bounded since all of $\mathcal{C}$ is contained in a ball of radius one! We define boundedness of a set in $(\mathcal{C}, \tau_w)$ as in any other linear topological space (see, for example, [12, p. 8]).

Definition 2.33. Let $A, B \subset \mathcal{C}$. Then we say that B absorbs A or that A is absorbed by B if there exists $\rho > 0$ in $\mathbb{R}$ such that $A \subset \alpha B$ for all $\alpha \in \mathbb{C}$ satisfying $|\alpha|_o \geq \rho$. We say that A is bounded in $(\mathcal{C}, \tau_w)$ if every zero-neighborhood absorbs A.

Proposition 2.34. *Let $A \subset \mathcal{C}$ be compact in $(\mathcal{C}, \tau_w)$. Then A is closed and bounded in $(\mathcal{C}, \tau_w)$.*

Proof. That A is closed in $(\mathcal{C}, \tau_w)$ follows from the fact that $(\mathcal{C}, \tau_w)$ is a Hausdorff topological space [11, p. 36].

Now we show that A is bounded in $(\mathcal{C}, \tau_w)$. We need to show that every zero-neighborhood in $(\mathcal{C}, \tau_w)$ absorbs A. So let U be a zero-neighborhood in $(\mathcal{C}, \tau_w)$. Then there exists $r > 0$ in $\mathbb{R}$ such that $B_w(0, r) \subset U$. Let $V = B_w(0, r/2)$; then $V + V \subset B_w(0, r) \subset U$, for if $z, \xi \in V$ then

$$\|z + \xi\|_{1/r} \leq \|z\|_{1/r} + \|\xi\|_{1/r} \leq \|z\|_{2/r} + \|\xi\|_{2/r} < \frac{r}{2} + \frac{r}{2} = r.$$

The family of sets $\{a + V : a \in A\}$ is an open cover of A in $(\mathcal{C}, \tau_w)$. By compactness of A, we can select a finite subcover: Thus, there exists $n \in \mathbb{N}$ and there exist $a_1, \ldots, a_n \in A$ such that $A \subset \bigcup_{j=1}^n (a_j + V)$. Since $V = B_w(0, r/2)$ is absorbing in $(\mathcal{C}, \tau_w)$, there exists $\rho > 1$ in $\mathbb{R}$ such that $a_j \in \alpha V$ for all $j \in \{1, \ldots, n\}$ and for all $\alpha \in \mathbb{C}$ satisfying $|\alpha|_o \geq \rho$. Thus for each $j = 1, \ldots, n$ we have that

$$a_j + V \subset a_j + \alpha V \subset \alpha V + \alpha V = \alpha(V + V) \subset \alpha U$$

all $\alpha \in \mathbb{C}$ satisfying $|\alpha|_o \geq \rho$. Hence

$$A \subset \bigcup_{j=1}^{n} (a_j + V) \subset \alpha U$$

for all $\alpha \in \mathbb{C}$ satisfying $|\alpha|_o \geq \rho$. Thus, U absorbs A. □

Proposition 2.35. $B_v(0, 1)$ *is not bounded in $(\mathcal{C}, \tau_w)$.*

Proof. It suffices to show that $B_v(0, 1)$ is not absorbed by $B_w(0, 1)$. That is, it suffices to show that, for all $\alpha \neq 0$ in $\mathbb{C}$, there exists $z \in B_v(0, 1)$ such that $z \notin \alpha B_w(0, 1)$. So let $\alpha \neq 0$ in $\mathbb{C}$ be given. Let $z = 2\alpha d$. Then $z \in B_v(0, 1)$ but $z \notin \alpha B_w(0, 1)$ since $\|z\|_1 = 2|\alpha|_o > |\alpha|_o$. □

Remark 2.36. Similarly, we can show that none of the balls $B_v(z_0, r)$, $B_v[z_0, r]$, $B_o(z_0, t)$, or $B_o[z_0, t]$ are bounded in $(\mathcal{C}, \tau_w)$ for all $z_0 \in \mathcal{C}$, $r > 0$ in $\mathbb{R}$ and $t > 0$ in $\mathcal{R}$.

Corollary 2.37. *None of the balls $B_v(z_0, r)$, $B_v[z_0, r]$, $B_o(z_0, t)$, or $B_o[z_0, t]$ are compact in $(\mathcal{C}, \tau_w)$ for all $z_0 \in \mathcal{C}$, $r > 0$ in $\mathbb{R}$ and $t > 0$ in $\mathcal{R}$.*

Proposition 2.38. *For all $r > 0$ in $\mathbb{R}$, $B_w[0, r]$ is closed but not bounded and hence not compact in $(\mathcal{C}, \tau_w)$. Thus, $(\mathcal{C}, \tau_w)$ is neither locally bounded nor locally compact.*

Proof. Let $\xi \in \mathcal{C} \setminus B_w[0, r]$. Then $\|\xi\|_{1/r} > r$. Let

$$t = \min\{\|\xi\|_{1/r} - r, r\}.$$

Then $t > 0$ in $\mathbb{R}$. We show that $B_w(\xi, t) \subset \mathcal{C} \setminus B_w[0, r]$: Let $z \in B_w(\xi, t)$ be given. Then $\|\xi - z\|_{1/t} = \|z - \xi\|_{1/t} < t$. It follows that

$$\begin{aligned}
\|z\|_{1/r} &\geq \|\xi\|_{1/r} - \|\xi - z\|_{1/r} \\
&\geq \|\xi\|_{1/r} - \|\xi - z\|_{1/t} \\
&> \|\xi\|_{1/r} - t \\
&\geq \|\xi\|_{1/r} - (\|\xi\|_{1/r} - r) = r.
\end{aligned}$$

This shows that $z \notin B_w[0, r]$ for all $z \in B_w(\xi, t)$ and hence $B_w(\xi, t) \subset \mathcal{C} \setminus B_w[0, r]$. It follows that $\mathcal{C} \setminus B_w[0, r]$ is open in $(\mathcal{C}, \tau_w)$, and hence $B_w[0, r]$ is closed in $(\mathcal{C}, \tau_w)$.

To show that $B_w[0, r]$ is not bounded in $(\mathcal{C}, \tau_w)$, it suffices to show that there exists a zero-neighborhood in $(\mathcal{C}, \tau_w)$ which does not absorb $B_w[0, r]$. Let $q \in \mathbb{Q}$ be such that

$$0 < q < \min\left\{\frac{r}{2}, \frac{1}{2r}\right\}.$$

We show that $B_w[0, r]$ is not absorbed by $B_w(0, q)$. Let $\alpha \neq 0$ in $\mathbb{C}$ be given. Let $z = 2\alpha q d^{1/q}$. Then

$$\|z\|_{1/q} = 2|\alpha|_o q > |\alpha|_o q; \text{ and hence } z \notin \alpha B_w(0, q).$$

However, since $0 < q < r/2$, it follows that $1/q > 2/r > 1/r$; and hence

$$\|z\|_{1/r} = 0 < r, \text{ so } z \in B_w[0, r].$$

□

Corollary 2.39. *For all $r > 0$ in $\mathbb{R}$, $B_w(0, r)$ is not bounded in $(\mathcal{C}, \tau_w)$.*

Remark 2.40. Since every p-normed space (with $0 < p \leq 1$) is locally bounded, we infer that there can be no p-norm (with $0 < p \leq 1$) that induces the topology τ_w on $\mathcal{C}$.

Using the results of Corollary 2.37 and Proposition 2.38 (or Corollary 2.39), we readily obtain the following result.

Corollary 2.41. *Let A be compact in $(\mathcal{C}, \tau_w)$. Then A has an empty interior in both $(\mathcal{C}, \tau_v)$ and $(\mathcal{C}, \tau_w)$; that is*

$$\begin{aligned}
int_v(A) &:= \{a \in A : \exists r > 0 \text{ in } \mathbb{R} \ni B_v(a, r) \subset A\} = \emptyset, \text{ and} \\
int_w(A) &:= \{a \in A : \exists r > 0 \text{ in } \mathbb{R} \ni B_w(a, r) \subset A\} = \emptyset.
\end{aligned}$$

Proposition 2.42. *Let $A \subset \mathcal{C}$ be bounded in $(\mathcal{C}, \tau_w)$. Then there exists $M > 0$ in $\mathbb{R}$ such that $\|z\|_{1/M} \leq M$ for all $z \in A$; that is, $A \subset B_w[0, M]$.*

Proof. Since A is bounded in $(\mathcal{C}, \tau_w)$, A is absorbed by every zero-neighborhood in $(\mathcal{C}, \tau_w)$. In particular, A is absorbed by $B_w(0, r)$ for some fixed $r > 0$ in $\mathbb{R}$. Thus, there exists $\alpha > 1$ in $\mathbb{R}$ such that $A \subset \alpha B_w(0, r)$. Hence $\|z\|_{1/r} < \alpha r$ for all $z \in A$. Let $M = \alpha r$. Then $M \in \mathbb{R}$ and $M > r > 0$. Thus, $0 < 1/M < 1/r$. Moreover, for all $z \in A$, we have that

$$\|z\|_{1/M} \leq \|z\|_{1/r} < \alpha r = M.$$

Hence $A \subset B_w[0, M]$. □

Remark 2.43. Proposition 2.38 shows that the converse of Proposition 2.42 is not true.

Remark 2.44. Convergence of sequences and series in $(\mathcal{R}, \tau_v)$, $(\mathcal{R}, \tau_w)$, $(\mathcal{C}, \tau_v)$ and $(\mathcal{C}, \tau_w)$ has been studied in detail in [2, 14, 17]. In particular, it is shown that $(\mathcal{R}, \tau_v)$ and $(\mathcal{C}, \tau_v)$ are Cauchy complete but $(\mathcal{R}, \tau_w)$ and $(\mathcal{C}, \tau_w)$ are not. For example, the sequence $(a_n)_{n\in\mathbb{N}}$, where $a_n = \sum_{j=1}^{n} d^{-j}/j$ for each $n \in \mathbb{N}$, is Cauchy in $(\mathcal{R}, \tau_w)$ (resp. in $(\mathcal{C}, \tau_w)$) but it does not converge in $(\mathcal{R}, \tau_w)$ (resp. in $(\mathcal{C}, \tau_w)$).

3. Analysis on $\mathcal{C}$

In this section we will define continuity and differentiability of a function from $A \subset \mathcal{C} \to \mathcal{C}$ at a point $z_0 \in A$ as well as on A. Then we will show that some basic results from classical complex analysis work in $\mathcal{C}$ as well but other fundamental results don't work due to the total disconnectedness of $(\mathcal{C}, \tau_v)$.

Definition 3.1. Let $A \subset \mathcal{C}$, let $f : A \to \mathcal{C}$ and let $z_0 \in A$ be given. Then we say that f is continuous at z_0 if for all $\epsilon > 0$ in $\mathbb{R}$ there exists $\delta > 0$ in $\mathbb{R}$ such that

$$z \in A \text{ and } |z - z_0| < \delta \Rightarrow |f(z) - f(z_0)| < \epsilon.$$

Moreover, we say that f is continuous on A if f is continuous at every $z \in A$.

Definition 3.2. Let $A \subset \mathcal{C}$ be open, let $f : A \to \mathcal{C}$ and let $z_0 \in A$ be given. Then we say that f is differentiable at z_0 if there exists a number $\xi \in \mathcal{C}$ such that for all $\epsilon > 0$ in $\mathbb{R}$ there exists $\delta > 0$ in $\mathbb{R}$ such that

$$z \in A \text{ and } 0 < |z - z_0| < \delta \Rightarrow \left| \frac{f(z) - f(z_0)}{z - z_0} - \xi \right| < \epsilon.$$

If this is case, we call the number ξ the derivative of f at z_0 and denote it by $f'(z_0)$.

Moreover, we say that f is differentiable on A if f is differentiable at every $z \in A$.

In the following, we will list basic results and rules about continuous and differentiable functions at a point or on a set in $\mathcal{C}$. We omit the proofs here as they are identical to those of the respective results in $\mathbb{C}$ or in any other metric space.

- Let $A \subset \mathcal{C}$, let $f : A \to \mathcal{C}$ and let $z_0 \in A$ be given. Then f is continuous at z_0 if and only if for any sequence (ξ_n) in A that converges to z_0 in $(\mathcal{C}, \tau_v)$, the sequence $(f(\xi_n))$ converges to $f(z_0)$ in $(\mathcal{C}, \tau_v)$.

- Let $A \subset \mathcal{C}$, let $f, g : A \to \mathcal{C}$ be continuous at $z_0 \in A$ (resp. on A), and let $\alpha \in \mathcal{C}$ be given. Then $f + \alpha g$ and $f \cdot g$ are continuous at z_0 (resp. on A).

- Let $A, B \subset \mathcal{C}$, let $f : A \to B$ be continuous at $z_0 \in A$ (resp. on A) and $g : B \to \mathcal{C}$ be continuous at $f(z_0)$ (resp. on B). Then $g \circ f$ is continuous at z_0 (resp. on A).

- Let $A \subset \mathcal{C}$ be open and let $f : A \to \mathcal{C}$ be differentiable at $z_0 \in A$ (resp. on A). Then f is continuous at z_0 (resp. on A).

- Let $A \subset \mathcal{C}$ be open, let $f, g : A \to \mathcal{C}$ be differentiable at $z_0 \in A$ (resp. on A), and let $\alpha \in \mathcal{C}$ be given. Then $f + \alpha g$ is differentiable at z_0 (resp. on A), with derivative
$$\begin{aligned} (f+\alpha g)'(z_0) &= f'(z_0) + \alpha g'(z_0) \\ (\text{resp. } (f+\alpha g)'(z) &= f'(z) + \alpha g'(z) \text{ for all } z \in A). \end{aligned}$$
- (Product Rule) Let $A \subset \mathcal{C}$ be open and let $f, g : A \to \mathcal{C}$ be differentiable at $z_0 \in A$ (resp. on A). Then $f \cdot g$ is differentiable at z_0 (resp. on A), with derivative
$$\begin{aligned} (f \cdot g)'(z_0) &= f'(z_0)g(z_0) + f(z_0)g'(z_0) \\ (\text{resp. } (f \cdot g)'(z) &= f'(z)g(z) + f(z)g'(z) \text{ for all } z \in A). \end{aligned}$$
- (Chain Rule) Let $A, B \subset \mathcal{C}$ be open, let $f : A \to B$ be differentiable at $z_0 \in A$ (resp. on A) and $g : B \to \mathcal{C}$ be differentiable at $f(z_0)$ (resp. on B). Then $g \circ f$ is differentiable at z_0 (resp. on A), with derivative
$$\begin{aligned} (g \circ f)'(z_0) &= g'(f(z_0))f'(z_0) \\ (\text{resp. } (g \circ f)'(z) &= g'(f(z))f'(z) \text{ for all } z \in A). \end{aligned}$$
- (Quotient Rule) Let $A \subset \mathcal{C}$ be open, let $f, g : A \to \mathcal{C}$ be differentiable at $z_0 \in A$, and let $g(z_0) \neq 0$. Then f/g is differentiable at z_0, with derivative
$$\left(\frac{f}{g}\right)'(z_0) = \frac{f'(z_0)g(z_0) - f(z_0)g'(z_0)}{g^2(z_0)}.$$
- (Differentiability and the Cauchy-Riemann Equations) Let $A \subset \mathcal{C}$ be open, let $f : A \to \mathcal{C}$, and let $z_0 \in A$ be given. Let $B = \{(x, y) \in \mathcal{R}^2 : x+iy \in A\}$ and write $f(z) = U(x, y)+iV(x, y)$ for $z = x+iy \in A$, with $U, V : B \to \mathcal{R}$. If f is differentiable at $z_0 = x_0 + iy_0$ in A then the partial derivatives of $U(x, y)$ and $V(x, y)$ exist at (x_0, y_0) and they satisfy the Cauchy-Riemann equations
$$\frac{\partial U}{\partial x}(x_0, y_0) = \frac{\partial V}{\partial y}(x_0, y_0) \text{ and } \frac{\partial U}{\partial y}(x_0, y_0) = -\frac{\partial V}{\partial x}(x_0, y_0).$$
  Conversely, if the Cauchy-Riemann equations hold and if U and V are differentiable as functions from $B \subset \mathcal{R}^2$ to $\mathcal{R}$ at (x_0, y_0), then f is differentiable at z_0, with derivative
$$\begin{aligned} f'(z_0) &= \frac{\partial U}{\partial x}(x_0, y_0) + i\frac{\partial V}{\partial x}(x_0, y_0) \\ &= \frac{\partial V}{\partial y}(x_0, y_0) - i\frac{\partial U}{\partial y}(x_0, y_0). \end{aligned}$$

In the following, we give an example of a function that is single-valued and infinitely often differentiable on the unit ball $B_v[0, 1]$ of $(\mathcal{C}, \tau_v)$ but whose Taylor series around any point $\xi \in B_v[0, 1]$ does not converge to the function at any $z \neq \xi$. Then we give another example of a function that is single-valued and differentiable on $B_v[0,1]$ but not twice differentiable at 0. These two examples are counterintuitive to what we are used to in classical Complex Analysis and indicate that more work needs to be done in order to develop a complete Analysis on the complex Levi-Civita field $\mathcal{C}$. Ongoing research aims at overcoming the difficulties arising from the total disconnectedness of $(\mathcal{C},\tau_v)$ and developing a meaningful analysis theory on the field. In particular, we will work on developing a Cauchy-like integration theory on $\mathcal{C}$ and then study under what conditions we can prove analogues of the core results of classical Complex Analysis such as the Cauchy Integral Theorem, the Cauchy Integral Formula and the Residue Theorem.

Example 3.3. Let $g : B_v[0,1] \to \mathcal{C}$ be given by

$$g(\xi)[q] = \xi[q/3] \text{ for all } q \geq 0 \text{ in } \mathbb{Q}.$$

Thus, given $\xi \in B_v[0,1]$, we can write $\xi = a_0 + \sum_{j=1}^{\infty} a_j d^{q_j}$, with $a_j \in \mathbb{C}$ for $j \geq 0$ and $0 < q_1 < q_2 < \cdots$; then $g(\xi) = a_0 + \sum_{j=1}^{\infty} a_j d^{3q_j}$.

We show first that g is (uniformly) differentiable on $B_v[0,1]$ with $g'(\xi) = 0$ for all $\xi \in B_v[0,1]$. So let $\epsilon > 0$ in $\mathbb{R}$ be given. Let

$$\delta = \min\{\epsilon, 1\}.$$

Then $\delta > 0$ in $\mathbb{R}$. Now let $z, \xi \in B_v[0,1]$ be such that $0 < |z-\xi| < \delta$. Then

$$g(z) - g(\xi) = g(z-\xi) \sim (z-\xi)^3, \text{ and hence } |g(z)-g(\xi)| = |z-\xi|^3.$$

It follows that

$$\left|\frac{g(z)-g(\xi)}{z-\xi} - 0\right| = \frac{|g(z)-g(\xi)|}{|z-\xi|} = |z-\xi|^2 < \delta^2 < \delta \leq \epsilon.$$

However, for all $\xi \in B_v[0,1]$ and for $z \neq \xi$ in $B_v[0,1]$, we have that

$$g(z) \neq \sum_{n=0}^{\infty} \frac{g^{(n)}(\xi)}{n!}(z-\xi)^n = g(\xi).$$

Example 3.4. Let $f : B_v[0,1] \to \mathcal{C}$ be given by

$$f(z) = \begin{cases} \frac{g(z)}{z} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0, \end{cases}$$

where g is the function of Example 3.3 above. Then

$$f'(0) = \lim_{h\to 0} \frac{f(h)}{h} = \lim_{h\to 0} \frac{g(h)}{h^2} = 0 \text{ since } \frac{g(h)}{h^2} \sim h \text{ for } 0 < |h| < 1;$$

and

$$f'(z) = -\frac{g(z)}{z^2} \text{ for } z \neq 0$$

using the Quotient Rule and the fact that $g'(z) = 0$.

Even though f is single-valued and (continuously) differentiable on $B_v[0,1]$, f is not twice differentiable at 0 since

$$\lim_{h\to 0} \frac{f'(h)-f'(0)}{h} = -\lim_{h\to 0} \frac{g(h)}{h^3}$$

does not exist. In fact, for $h=\sum_{j=1}^{\infty} a_j d^{q_j}$ with $a_1 \neq 0$ and $0<q_1<q_2<\cdots$ $(0<|h|<1)$, we have that

$$\frac{f'(h)-f'(0)}{h} = -\frac{g(h)}{h^3} = -\frac{a_1 d^{3q_1}+a_2 d^{3q_2}+\cdots}{\left(a_1 d^{q_1}+a_2 d^{q_2}+\cdots\right)^3} \approx -\frac{1}{a_1^2}$$

has no limit as $h \to 0$ (that is, as $q_1 \to \infty$.)

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