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Calculus on a non-Archimedean field extension of the real numbers: inverse function theorem, intermediate value theorem and mean value theorem

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Abstract. In this paper, we introduce the concept of weakly locally uniformly differentiable functions (WLUD) on $\mathcal{N}$, a non-Archimedean field extension of the real numbers that is real closed and Cauchy complete in the topology induced by the order. We show that WLUD functions are C^1 and they form an $\mathcal{N}$-algebra that is closed under composition and contains all polynomial functions.

We formulate and prove a version of the inverse function theorem as well as a local intermediate value theorem for these functions. Then we generalize the WLUD concept to higher orders of differentiability and study WLUD^n functions at a point or on a subset of $\mathcal{N}$. In particular, we study the properties of WLUD^2 functions and we formulate and prove a local mean value theorem for such functions.

1. Introduction

We start this section by reviewing some basic terminology and facts about non-Archimedean fields. So let F be an ordered non-Archimedean field extension of $\mathbb{R}$. We introduce the following terminology.

Definition 1 ($\sim$, $\approx$, $\ll$, S_F, λ). For $x, y \in F^* := F \setminus \{0\}$, we say that x is of the same order as y and write $x \sim y$ if there exist $n, m \in \mathbb{N}$ such that $n|x| > |y|$ and $m|y| > |x|$, where $|\cdot|$ denotes the ordinary absolute value on F: $|x| = \max\{x, -x\}$. For nonnegative $x, y \in F$, we say that x is infinitely smaller than y and write $x \ll y$ if $nx < y$ for all $n \in \mathbb{N}$, and we say that x is infinitely small if $x \ll 1$ and x is finite if $x \sim 1$; finally, we say that x is approximately equal to y and write $x \approx y$ if $x \sim y$ and $|x - y| \ll |x|$. We also set $\lambda(x) = [x]$, the class of x under the equivalence relation $\sim$.

The set of equivalence classes S_F (under the relation $\sim$) is naturally endowed with an addition via $[x] + [y] = [x \cdot y]$ and an order via $[x] < [y]$ if $|y| \ll |x|$ (or $|x| \gg |y|$), both of which are readily checked to be well-defined. Note that we use $+$ instead of $\cdot$ for the operation in S_F because, for the fields discussed in this paper, S_F is isomorphic to an additive subgroup of $\mathbb{R}$. It follows that $(S_F, +, <)$ is an ordered group, often referred to as the Hahn group or skeleton group, whose neutral element is $[1]$, the class of 1. It follows from the above that the projection λ from F^* to S_F is a valuation.

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The theorem of Hahn [**2**] provides a complete classification of non-Archimedean extensions of $\mathbb{R}$ in terms of their skeleton groups. In fact, invoking the axiom of choice it is shown that the elements of any such ordered field F can be written as (generalized) formal power series (also called Hahn series) over its skeleton group S_F with real coefficients, and the set of appearing exponents forms a well-ordered subset of S_F. That is, for all $x \in F$, we have

$$x = \sum_{q \in S_F} a_q d^q; \tag{1.1}$$

with $a_q \in \mathbb{R}$ for all q, d a positive infinitely small element of F, and the support of x, given by

$$\operatorname{supp}(x) := \{q \in S_F : a_q \neq 0\},$$

forming a well-ordered subset of S_F. With the representation of elements of F as in Equation (1.1) it follows that, for $x \neq 0$ in F,

$$\lambda(x) = \min\left(\operatorname{supp}(x)\right),$$

which exists since $\operatorname{supp}(x)$ is well-ordered. Moreover, we set $\lambda(0) = \infty$.

Addition, multiplication and order on the Hahn series are defined as follows. Given $x = \sum_{q \in \operatorname{supp}(x)} a_q d^q$ and $y = \sum_{t \in \operatorname{supp}(y)} b_t d^t$, then

$$x + y = \sum_{r \in \operatorname{supp}(x) \cup \operatorname{supp}(y)} (a_r + b_r) d^r; \text{ and}$$

$$x \cdot y = \sum_{r \in \operatorname{supp}(x) \oplus \operatorname{supp}(y)} \left(\sum_{\substack{q \in \operatorname{supp}(x),\, t \in \operatorname{supp}(y) \\ q + t = r}} a_q \cdot b_t \right) d^r. \tag{1.2}$$

Note that, since $\operatorname{supp}(x)$ and $\operatorname{supp}(y)$ are well-ordered, only finitely many terms contribute to the sum

$$\sum_{\substack{q \in \operatorname{supp}(x),\, t \in \operatorname{supp}(y) \\ q + t = r}} a_q \cdot b_t,$$

in Equation (1.2), for each $r \in \operatorname{supp}(x) \oplus \operatorname{supp}(y)$.

Given a nonzero $x = \sum_{q \in \operatorname{supp}(x)} a_q d^q$, then $x > 0$ if and only if $a_{\lambda(x)} > 0$.

From general properties of formal power series fields [**6**, **8**], it follows that if S_F is divisible then F is real closed; that is, every positive element of F is a square in F and every polynomial of odd degree over F has at least one root in F. For a general overview of the algebraic properties of formal power series fields, we refer to the comprehensive overview by Ribenboim [**9**], and for an overview of the related valuation theory the book by Krull [**3**]. A thorough and complete treatment of ordered structures can also be found in [**7**].

Throughout this paper, we will denote by $\mathcal{N}$ any totally ordered non-Archimedean field extension of $\mathbb{R}$ that is real closed and complete in the order topology and whose skeleton group $S_{\mathcal{N}}$ is Archimedean, i.e. a subgroup of $\mathbb{R}$. The coefficient a_q of the qth power in the Hahn representation of a given x will be denoted by $x[q]$, and hence the number d is given by $d[1] = 1$ and $d[q] = 0$ for $q \neq 1$. It is easy to check that, for $q \in S_{\mathcal{N}}$, $0 < d^q \ll 1$ if and only if $q > 0$, and $d^q \gg 1$ if and only if $q < 0$; moreover, $x \approx x[\lambda(x)] d^{\lambda(x)}$ for all $x \neq 0$.

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The smallest such field $\mathcal{N}$ is the Levi-Civita field $\mathcal{R}$, first introduced in [**4**, **5**]. In this case $S_{\mathcal{R}} = \mathbb{Q}$, and for any element $x \in \mathcal{R}$, $\mathrm{supp}(x)$ is a left-finite subset of $\mathbb{Q}$, i.e. below any rational bound r there are only finitely many exponents in the Hahn representation of x. The Levi-Civita field $\mathcal{R}$ is of particular interest because of its practical usefulness. Since the supports of the elements of $\mathcal{R}$ are left-finite, it is possible to represent these numbers on a computer [**1**]. Having infinitely small numbers allows for many computational applications; one such application is the computation of derivatives of real functions representable on a computer [**13**], where both the accuracy of formula manipulators and the speed of classical numerical methods are achieved. For a review of the Levi-Civita field $\mathcal{R}$, see [**12**, **14**] and references therein.

In the wider context of valuation theory, it is interesting to note that the topology induced by the order on $\mathcal{N}$ is the same as that introduced via the valuation λ, as shown in Remark 1 below. It follows therefore that the field $\mathcal{N}$ is just a special case of the class of fields discussed in [**11**].

Remark 1. *The mapping $\Lambda : \mathcal{N} \times \mathcal{N} \to \mathbb{R}$, given by $\Lambda(x, y) = \exp(-\lambda(x - y))$, is an ultrametric distance (and hence a metric); the valuation topology it induces is equivalent to the order topology (we will use τ_v to denote either one of the two topologies in this paper). For if A is an open set in the order topology and $a \in A$, then there exists $r > 0$ in $\mathcal{N}$ such that, for all $x \in \mathcal{N}$, $|x - a| < r \Rightarrow x \in A$. Let $l = \exp(-\lambda(r))$, then we also have that, for all $x \in \mathcal{N}$, $\Lambda(x, a) < l \Rightarrow x \in A$; and hence A is open with respect to the valuation topology. The other direction of the equivalence of the topologies follows analogously.*

It follows from Remark 1 that $\mathcal{N}$ which is complete in the order topology is also complete in the valuation topology induced by the ultrametric Λ.

Remark 2. *Contrary to the field $\mathbb{R}^*$ of Nonstandard Analysis* [**10**, **18**], *the field $\mathcal{N}$ is an ordered field extension of the field of real numbers $\mathbb{R}$; and the enmbedding of $\mathbb{R}$ in $\mathcal{N}$ is compatible with the orders in $\mathbb{R}$ and $\mathcal{N}$. While in Nonstandard Analysis there is a generally valid transfer principle that allows the transformation of known results of conventional analysis, here all relevant calculus theorems are developed separately. Moreover, besides being non-Archimedeanly valued, the fact that the Levi-Civita field $\mathcal{R}$ has a total order (which is also non-Archimedean) gives the field a richer structure, thus opening up new possibilities of study, like monotonicity, which are not available in other non-Archimedean valued fields like the p-Adic fields for example* [**11**]. *This makes $\mathcal{N}$ an outstanding example, worthy to be studied in detail in its own right.*

The following results were proved in [**17**]; they show that the topological structure of $\mathcal{N}$ is different from that of $\mathbb{R}$ or $\mathbb{C}$, and that makes doing Calculus on the field more difficult.

- $(\mathcal{N}, \tau_v)$ is a totally disconnected topological space. It is Hausdorff and nowhere locally compact. There are no countable bases. The topology induced to $\mathbb{R}$ is the discrete topology. As an immediate consequence of the fact that $(\mathcal{N}, \tau_v)$ is totally disconnected, it follows that, for any $x_0 \in \mathcal{N}$, the connected component of x_0 is $\{x_0\}$; moreover, the topology is zero-dimensional, that is, there is a base of clopen sets for the topology.
- If we view $\mathcal{N}$ as an infinite dimensional vector space over $\mathbb{R}$ then τ_v is not a vector topology; that is, $(\mathcal{N}, \tau_v)$ is not a linear topological space.

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- If A is compact in $(\mathcal{N}, \tau_v)$ then A is closed and bounded and it has an empty interior in $(\mathcal{N}, \tau_v)$, that is,

  $$\mathrm{int}(A) := \{a \in A : \exists r > 0 \text{ in } \mathcal{N} \ni (a-r, a+r) \subset A\} = \emptyset.$$

  The converse is not true: the set $A = [0,1] \cap \mathbb{Q}$ is a (countably infinite) closed and bounded subset of $\mathcal{N}$ with an empty interior; but A is not compact in $(\mathcal{N}, \tau_v)$ [**17**].
- Given a sequence (x_n) of elements of $\mathcal{N}$, the series $\sum_{n=1}^{\infty} x_n$ converges if and only if the sequence (x_n) converges to zero.

In [**17**] we studied the properties of locally uniformly differentiable (LUD) functions on $\mathcal{N}$. In particular, we showed that this class of functions is closed under addition, multiplication and composition of functions. Then we stated and proved local versions of the inverse function theorem and the intermediate value theorem for $\mathcal{N}$-valued LUD functions on $\mathcal{N}$. The stronger condition (local uniform differentiability) on the function than that of the real case was needed for the proofs of both theorems because of the total disconnectedness of the field $\mathcal{N}$ in the order topology. Then in [**15**], we generalized the definition of local uniform differentiability to any order. Then we studied the properties of n-times locally uniformly differentiable (LUD^n) functions and we formulated and proved a local mean value theorem for $\mathcal{N}$-valued functions that are LUD^2 at a point of $\mathcal{N}$.

In this paper, we introduce a new smoothness criterion which we call weak local uniform differentiability (WLUD) which is strictly weaker than local uniform differentiability and strictly stronger than continuous differentiability (C^1), we study the properties of $\mathcal{N}$-valued WLUD and WLUD^n functions and we show that this weaker criterion is sufficient to get all the nice calculus results obtained in [**15, 17**].

2. WLUD Functions

Definition 2. Let $A \subseteq \mathcal{N}$ be open, let $f : A \to \mathcal{N}$, and let $x_0 \in A$ be given. We say that f is weakly locally uniformly differentiable (abbreviated as WLUD) at x_0 if f is differentiable in a neighbourhood of x_0 in A and if for every $\epsilon > 0$ in $\mathcal{N}$ there exists $\delta > 0$ in $\mathcal{N}$ such that for every $x, y \in (x_0 - \delta, x_0 + \delta) \cap A$ we have that $|f(y) - f(x) - f'(x)(y-x)| < \epsilon\, |y - x|$. Moreover, we say that f is WLUD on A if f is WLUD at every point in A.

One may notice, that the definition of WLUD is very similar to that of LUD in [**15, 17**], and indeed it is easy to see that LUD implies WLUD. However, as we will prove later, the two concepts are not equivalent. Nonetheless, similar to the case with LUD functions, we get that the class of WLUD functions is contained in the class of C^1 functions, and is an $\mathcal{N}$-algebra that is closed under composition of functions.

Notation 1. Throughout the rest of the paper, we will use $B(x_0, r)$ to denote the open interval $(x_0 - r, x_0 + r)$, for $x_0 \in \mathcal{N}$ and $r > 0$ in $\mathcal{N}$.

Proposition 1. *Let $A \subseteq \mathcal{N}$ be open and let $f : A \to \mathcal{N}$ be WLUD at $x_0 \in A$. Then f is C^1 at x_0.*

Proof. Let $\epsilon > 0$ in $\mathcal{N}$ be given. Then there exists $\delta > 0$ in $\mathcal{N}$ such that, for every $x, y \in B(x_0, \delta) \cap A$, we have that

$$|f(y) - f(x) - f'(x)(y-x)| < \frac{\epsilon}{2}\, |y - x|\,.$$

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It follows that, for all $x \in B(x_0, \delta) \cap A$, we have that

$$\begin{aligned} |f'(x) - f'(x_0)| &\le \left| f'(x) - \frac{f(x_0) - f(x)}{x_0 - x} \right| + \left| \frac{f(x) - f(x_0)}{x - x_0} - f'(x_0) \right| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

□

PROPOSITION 2. *Let $f, g : A \to \mathcal{N}$ be WLUD at $x_0 \in A$ and let $\alpha \in \mathcal{N}$ be given. Then $(f + \alpha g)$ is WLUD at x_0.*

PROOF. Without loss of generality, we may assume $\alpha \neq 0$. Let $\epsilon > 0$ in $\mathcal{N}$ be given. Then there exists $\delta > 0$ in $\mathcal{N}$ such that, for every $x, y \in B(x_0, \delta) \cap A$, we have that

$$|f(y) - f(x) - f'(x)(y - x)| < \frac{\epsilon}{2}|y - x|$$

and

$$|g(y) - g(x) - g'(x)(y - x)| < \frac{\epsilon}{2|\alpha|}|y - x|.$$

Hence, for every $x, y \in B(x_0, \delta) \cap A$, we have that

$$\begin{aligned} &|(f+\alpha g)(y) - (f + \alpha g)(x) - (f' + \alpha g')(x)(y - x)| \\ &\quad \le |f(y) - f(x) - f'(x)(y - x)| + |\alpha|\, |g(y) - g(x) - g'(x)(y - x)| \\ &\quad < \frac{\epsilon}{2}|y - x| + |\alpha| \frac{\epsilon}{2|\alpha|}|y - x| \\ &\quad = \epsilon|y - x|. \end{aligned}$$

□

PROPOSITION 3. *Let $f, g : A \to \mathcal{N}$ be WLUD at $x_0 \in A$. Then fg is WLUD at x_0.*

PROOF. Let $\epsilon > 0$ in $\mathcal{N}$ be given. By proposition 1, f and g are C^1 at x_0, and so there exists a $\delta_c > 0$ in $\mathcal{N}$ such that $|f(x) - f(x_0)| < 1$, $|g(x) - g(x_0)| < 1$, and $|f'(x) - f'(x_0)| < 1$ on $B(x_0, \delta_c) \cap A$. Moreover, there exist $\delta_f, \delta_g, \delta_0 > 0$ in $\mathcal{N}$ such that

$$|f(y) - f(x) - f'(x)(y - x)| < \frac{\epsilon}{3(|g(x_0)| + 1)}\, |y - x|$$

if $x, y \in B(x_0, \delta_f) \cap A$;

$$|g(y) - g(x) - g'(x)(y - x)| < \frac{\epsilon}{3(|f(x_0)| + 1)}\, |y - x|$$

if $x, y \in B(x_0, \delta_g) \cap A$; and

$$|g(y) - g(x)| < \frac{\epsilon}{3(|f'(x_0)| + 1)}$$

if $x, y \in B(x_0, \delta_0) \cap A$.

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Let $\delta = \min\{\delta_c, \delta_f, \delta_g, \delta_0\}$. Then it follows that, for every $x, y \in B(x_0, \delta) \cap A$, we have that

$$\begin{aligned}
&\left| f(y)g(y) - f(x)g(x) - \Big(f'(x)g(x) + f(x)g'(x)\Big)(y-x) \right| \\
&\leq |f(y)g(y) - f(x)g(y) - f'(x)g(y)(y-x)| \\
&\qquad + |f(x)g(y) - f(x)g(x) - f(x)g'(x)(y-x)| \\
&\qquad + |f'(x)g(y)(y-x) - f'(x)g(x)(y-x)| \\
&= |g(y)|\,|f(y) - f(x) - f'(x)(y-x)| \\
&\qquad + |f(x)|\,|g(y) - g(x) - g'(x)(y-x)| \\
&\qquad + |f'(x)|\,|g(y) - g(x)|\,|y-x| \\
&< \frac{|g(y)|}{3(|g(x_0)|+1)}\epsilon|y-x| + \frac{|f(x)|}{3(|f(x_0)|+1)}\epsilon|y-x| + \frac{|f'(x)|}{3(|f'(x_0)|+1)}\epsilon|y-x| \\
&< \epsilon|y-x|.
\end{aligned}$$

□

PROPOSITION 4. *Let $g : A \to B$ be WLUD at $x_0 \in A$ and $f : B \to \mathcal{N}$ be WLUD at $g(x_0) \in B$. Then $f \circ g : A \to \mathcal{N}$ is WLUD at x_0.*

PROOF. Let $\epsilon > 0$ in $\mathcal{N}$ be given. As g is WLUD at x_0, and f is WLUD at $g(x_0)$, there exist $\delta_1, \delta_2 > 0$ in $\mathcal{N}$ such that

$$|f(y) - f(x) - f'(x)(y-x)| < \frac{\epsilon}{2\left(1 + |g'(x_0)|\right)}|y-x|$$

for every $x, y \in B \cap B(g(x_0), \delta_1)$; and

$$|g(y) - g(x) - g'(x)(y-x)| < \frac{\epsilon}{2\Big(\left|f'\big(g(x_0)\big)\right| + 1\Big)}|y-x|$$

for every $x, y \in A \cap B(x_0, \delta_2)$.

Moreover, as f is C^1 at $g(x_0)$ and g is C^1 at x_0, there exists a $\delta_3 > 0$ in $\mathcal{N}$ such that for every $x, y \in A \cap B(x_0, \delta_3)$ we have that $|g(y) - g(x)| < \delta_1$ and $|f'(g(x)) - f'(g(x_0))| < 1$. Finally, since g is WLUD at x_0, there exists $\delta_4 > 0$ in $\mathcal{N}$ such that for every $x, y \in A \cap B(x_0, \delta_4)$ we have that

$$|g(y) - g(x)| < \left(1 + |g'(x_0)|\right)|y-x|.$$

Let $\delta = \min\{\delta_1, \delta_2, \delta_3, \delta_4\}$. Then for every $x, y \in A \cap B(x_0, \delta)$ we have that

$$\begin{aligned}
&\left|f\big(g(y)\big) - f\big(g(x)\big) - g'(x)f'\big(g(x)\big)(y-x)\right| \\
&\leq \left| f\big(g(y)\big) - f\big(g(x)\big) - f'\big(g(x)\big)\Big(g(y) - g(x)\Big) \right| \\
&\qquad + \left|f'\big(g(x)\big)\right| |g(y) - g(x) - g'(x)(y-x)| \\
&< \frac{\epsilon}{2\left(1 + |g'(x_0)|\right)}\,|g(y) - g(x)| + \frac{\left|f'\big(g(x)\big)\right|}{2\Big(\left|f'\big(g(x_0)\big)\right| + 1\Big)}\epsilon|y-x| \\
&< \epsilon|y-x|.
\end{aligned}$$

□

In the following, we will extend the WLUD concept to higher orders of differentiability and we will define WLUD^n analogously to how LUD^n was defined in [**15**].

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Definition 3. Let $A \subseteq \mathcal{N}$ be open, let $f : A \to \mathcal{N}$, let $x_0 \in A$, and let $n \in \mathbb{N}$ be given. We say that f is WLUD^n at x_0 if f is n times differentiable in a neighbourhood of x_0 in A and if for every $\epsilon > 0$ in $\mathcal{N}$ there exists $\delta > 0$ in $\mathcal{N}$ such that for every $x, y \in B(x_0, \delta) \cap A$ we have that

$$\left| f(y) - \sum_{k=0}^{n} \frac{f^{(k)}(x)}{k!} (y-x)^k \right| < \epsilon \left| y - x \right|^n .$$

Moreover, we say that f is WLUD^n on A if f is WLUD^n at every point in A.

Remark 3. *If we use this definition to make an analogous concept of WLUD^0, we get the condition that for every $\epsilon > 0$ in $\mathcal{N}$ there exists $\delta > 0$ in $\mathcal{N}$ such that for every $x, y \in B(x_0, \delta) \cap A$ we have that $|f(y) - f(x)| < \epsilon$, which can be fairly easily seen to be an equivalent statement to that of f being continuous at x_0.*

Just as it is the case for LUD^n, in the real case WLUD^n is equivalent to C^n. Moreover, as we will see (in Proposition 5 below), this definition of WLUD^n implies WLUD^{n-1} (for $n \in \mathbb{N}$).

Lemma 1. *Let $A \subseteq \mathcal{N}$ be open and let $f : A \to \mathcal{N}$ be WLUD^n at $x_0 \in A$. Then $f^{(n)}$ is locally bounded at x_0; that is, there exist a neighborhood U of x_0 in A and an $M > 0$ in $\mathcal{N}$ such that, for every $x \in U$, we have that $\left| f^{(n)}(x) \right| \leq M$.*

Proof. As f is WLUD^n at x_0, there exists $\delta_1 > 0$ in $\mathcal{N}$ such that $B(x_0, \delta_1) \subset A$ and, for every $x, y \in B(x_0, \delta_1)$, we have that

$$\left| f(y) - \sum_{k=0}^{n} \frac{f^{(k)}(x)}{k!} (y-x)^k \right| < \left| y - x \right|^n .$$

For each $k \in \{0, 1, \dots, n-1\}$, $f^{(k)}$ is continuous at x_0, and so there exist a neighborhood V_k of x_0 in A and a number $M_k > 0$ in $\mathcal{N}$ such that for every $x \in V_k$ we have that $\left| f^{(k)}(x) \right| \leq M_k$. Let $\delta > 0$ in $\mathcal{N}$ be such that $B(x_0, \delta) \subseteq B(x_0, \delta_1) \cap \left(\cap_{k=0}^{n-1} V_k \right)$ and let $U = B(x_0, \delta)$. Now let

$$M = n! \left(\frac{2}{\delta} \right)^n \left(\left(\frac{\delta}{2} \right)^n + M_0 + \sum_{k=0}^{n-1} \frac{M_k}{k!} \left(\frac{\delta}{2} \right)^k \right),$$

and let $x \in U$ be given. Choose

$$y = \begin{cases} x + \frac{\delta}{2} & \text{if } x + \frac{\delta}{2} \in B(x_0, \delta) \\ x - \frac{\delta}{2} & \text{otherwise.} \end{cases}$$

Then we have that $x, y \in U \subseteq B(x_0, \delta_1)$ and thus,

$$\begin{aligned} \left| f^{(n)}(x) \right| &\leq \frac{n!}{|y-x|^n} \left(\left| f(y) - \sum_{k=0}^{n} \frac{f^{(k)}(x)}{k!} (y-x)^k \right| + |f(y)| + \sum_{k=0}^{n-1} \left| \frac{f^{(k)}(x)}{k!} (y-x)^k \right| \right) \\ &< \frac{n!}{|y-x|^n} \left(|y-x|^n + M_0 + \sum_{k=0}^{n-1} \frac{M_k}{k!} |y-x|^k \right) \\ &= n! \left(\frac{2}{\delta} \right)^n \left(\left(\frac{\delta}{2} \right)^n + M_0 + \sum_{k=0}^{n-1} \frac{M_k}{k!} \left(\frac{\delta}{2} \right)^k \right) = M. \end{aligned}$$

□

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Proposition 5. *Let $A \subseteq \mathcal{N}$ be open and let $f : A \to \mathcal{N}$ be WLUD^n at $x_0 \in A$ for some $n \in \mathbb{N}$. Then f is WLUD^{n-1} at x_0.*

Proof. By Lemma 1, there exists a neighborhood U of x_0 in A such that $f^{(n)}$ is locally bounded by some $M > 0$ on U. Let $\epsilon > 0$ in $\mathcal{N}$ be given. As f is WLUD^n at x_0, there exists $\delta_1 > 0$ in $\mathcal{N}$ such that, for every $x, y \in B(x_0, \delta_1)$, we have that

$$\left| f(y) - \sum_{k=0}^{n} \frac{f^{(k)}(x)}{k!}(y-x)^k \right| < \frac{\epsilon}{2} |y-x|^n .$$

Moreover, we may assume without loss of generality, that $B(x_0, \delta_1) \subseteq U$. Let $\delta = \min\{\delta_1, 1, n!\epsilon/(2M)\}$. Then, for every $x, y \in B(x_0, \delta)$, we have that

$$\begin{aligned}
\left| f(y) - \sum_{k=0}^{n-1} \frac{f^{(k)}(x)}{k!}(y-x)^k \right| &\le \left| f(y) - \sum_{k=0}^{n} \frac{f^{(k)}(x)}{k!}(y-x)^k \right| \\
&\quad + \left| \frac{f^{(n)}(x)}{n!}(y-x)^n \right| \\
&< \frac{\epsilon}{2} |y-x|^n + \left| \frac{f^{(n)}(x)}{n!} \right| \delta |y-x|^{n-1} \\
&\le \frac{\epsilon}{2} |y-x| \, |y-x|^{n-1} + \frac{M}{n!} \frac{n!\epsilon}{2M} |y-x|^{n-1} \\
&< \epsilon |y-x|^{n-1} .
\end{aligned}$$

□

Proposition 6. *Let $A \subseteq \mathcal{N}$ be open and let $f : A \to \mathcal{N}$ be WLUD^2 at $x_0 \in A$. Then f is C^2 at x_0.*

Proof. Let $\epsilon > 0$ in $\mathcal{N}$ be given. Then there exists $\delta_1 > 0$ in $\mathcal{N}$ such that, for every $x, y \in B(x_0, \delta_1)$, we have that

$$\left| f(y) - f(x) - f'(x)(y-x) - \frac{1}{2} f''(x)(y-x)^2 \right| < \frac{\epsilon}{6}(y-x)^2.$$

As f' is differentiable at x_0, there exists $\delta_2 > 0$ in $\mathcal{N}$ such that, for every $x \in A$ satisfying $0 < |x - x_0| < \delta_2$, we have that

$$\left| \frac{f'(x) - f'(x_0)}{x - x_0} - f''(x_0) \right| < \frac{\epsilon}{6}.$$

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Let $\delta = \min\{\delta_1, \delta_2\}$. Then it follows that if $0 < |x - x_0| < \delta$ then we have that

$$\begin{aligned}
|f''(x) - f''(x_0)| &\leq 2\left|\frac{1}{2}f''(x) + \frac{f'(x)}{x_0 - x} + \frac{f(x) - f(x_0)}{(x_0 - x)^2}\right| \\
&\quad + 2\left|\frac{f'(x) - f'(x_0)}{x - x_0} - f''(x_0)\right| \\
&\quad + 2\left|\frac{1}{2}f''(x_0) + \frac{f'(x_0)}{x - x_0} + \frac{f(x_0) - f(x)}{(x - x_0)^2}\right| \\
&= 2\left|\frac{f(x_0) - f(x)}{(x_0 - x)^2} - \frac{f'(x)}{x_0 - x} - \frac{1}{2}f''(x)\right| \\
&\quad + 2\left|\frac{f'(x) - f'(x_0)}{x - x_0} - f''(x_0)\right| \\
&\quad + 2\left|\frac{f(x) - f(x_0)}{(x - x_0)^2} - \frac{f'(x_0)}{x - x_0} - \frac{1}{2}f''(x_0)\right| \\
&< 2\frac{\epsilon}{6} + 2\frac{\epsilon}{6} + 2\frac{\epsilon}{6} = \epsilon.
\end{aligned}$$

□

PROPOSITION 7. *Let $A \subseteq \mathcal{N}$ be open and let $f : A \to \mathcal{N}$ be WLUD^2 at $x_0 \in A$. Then f' is WLUD at x_0.*

PROOF. Let $\epsilon > 0$ in $\mathcal{N}$ be given. As f is WLUD^2 at x_0, there exists $\delta_1 > 0$ in $\mathcal{N}$ such that $B(x_0, \delta_1) \subset A$ and, for any distinct $x, y \in B(x_0, \delta_1)$, we have that

$$\left|\frac{f(y) - f(x)}{y - x} - f'(x) - \frac{1}{2}f''(x)(y - x)\right| < d\epsilon|y - x|.$$

Moreover, by Proposition 6, f is C^2 at x_0, and hence there exists $\delta_2 > 0$ in $\mathcal{N}$ such that $B(x_0, \delta_2) \subset A$ and, for every $x, y \in B(x_0, \delta_2)$ we have that

$$|f''(y) - f''(x)| \leq |f''(y) - f''(x_0)| + |f''(x_0) - f''(x)| < d\epsilon.$$

Let $\delta = \min\{\delta_1, \delta_2\}$. Then it follows that for distinct $x, y \in B(x_0, \delta)$ we have that

$$\begin{aligned}
|f'(y) - f'(x) - f''(x)(y - x)| &\leq \left|\frac{f(y) - f(x)}{y - x} - f'(x) - \frac{1}{2}f''(x)(y - x)\right| \\
&\quad + \left|\frac{1}{2}f''(y)(x - y) + f'(y) - \frac{f(y) - f(x)}{y - x}\right| \\
&\quad + \frac{1}{2}\left|f''(y) - f''(x)\right||y - x| \\
&< d\epsilon|y - x| + d\epsilon|y - x| + \frac{1}{2}d\epsilon|y - x| < \epsilon|y - x|.
\end{aligned}$$

□

Just as with the $n = 1$ case, WLUD^n functions form an $\mathcal{N}$-algebra that is closed under composition, for any $n \in \mathbb{N}$. As this paper will focus primarily on functions that are WLUD^2, we mainly present the proofs for the $n = 2$ case here. The proofs for the general case are similar. We start with the following proposition whose proof (for any $n \geq 2$) is very similar to that of the case $n = 1$ (Proposition 2) and will thus be omitted here.

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Proposition 8. *Let $A \subseteq \mathcal{N}$ be open, let $f, g : A \to \mathcal{N}$ be WLUD^n at $x_0 \in A$ and let $\alpha \in \mathcal{N}$ be given. Then $(f + \alpha g)$ is WLUD^n at x_0.*

Proposition 9. *Let $A \subseteq \mathcal{N}$ be open and let $f, g : A \to \mathcal{N}$ be WLUD^2 at $x_0 \in A$. Then fg is WLUD^2 at x_0.*

Proof. Let $\epsilon > 0$ in $\mathcal{N}$ be given. By proposition 6, f and g are C^2 at x_0, and so there exists a $\delta_c > 0$ in $\mathcal{N}$ such that $B(x_0, \delta_c) \subset A$ and $|f(x) - f(x_0)| < 1$, $|g(x) - g(x_0)| < 1$, $|f'(x) - f'(x_0)| < 1$, and $|f''(x) - f''(x_0)| < 1$ on $B(x_0, \delta_c)$. Moreover, there exist $\delta_f, \delta_g, \delta_0, \delta_1 > 0$ in $\mathcal{N}$ such that

$$\left| f(y) - f(x) - f'(x)(y - x) - \frac{1}{2} f''(x)(y - x)^2 \right| < \frac{\epsilon}{4(|g(x_0)| + 1)} (y - x)^2$$

if $x, y \in B(x_0, \delta_f)$;

$$\left| g(y) - g(x) - g'(x)(y - x) - \frac{1}{2} g''(x)(y - x)^2 \right| < \frac{\epsilon}{4(|f(x_0)| + 1)} (y - x)^2$$

if $x, y \in B(x_0, \delta_g)$;

$$|g(y) - g(x) - g'(x)(y - x)| < \frac{\epsilon}{4(|f'(x_0)| + 1)} |y - x|$$

if $x, y \in B(x_0, \delta_0)$; and

$$|g(y) - g(x)| < \frac{\epsilon}{2(|f''(x_0)| + 1)}$$

if $x, y \in B(x_0, \delta_1)$.

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Let $\delta = \min\{\delta_c, \delta_f, \delta_g, \delta_0, \delta_1\}$. Then it follows that, for every $x, y \in B(x_0, \delta)$, we have that

$$\begin{aligned}
&\left|(fg)(y) - (fg)(x) - (fg)'(x)(y-x) - \frac{1}{2}(fg)''(x)(y-x)^2\right| \\
&= \Big| f(y)g(y) - f(x)g(x) - \Big(f'(x)g(x) + f(x)g'(x)\Big)(y-x) \\
&\qquad - \frac{1}{2}\Big(f''(x)g(x) + 2f'(x)g'(x) + f(x)g''(x)\Big)(y-x)^2 \Big| \\
&\le \left| f(y)g(y) - f(x)g(y) - f'(x)g(y)(y-x) - \frac{1}{2}f''(x)g(y)(y-x)^2 \right| \\
&\qquad + \left| f(x)g(y) - f(x)g(x) - f(x)g'(x)(y-x) - \frac{1}{2}f(x)g''(x)(y-x)^2 \right| \\
&\qquad + \left| f'(x)g(y)(y-x) - f'(x)g(x)(y-x) - f'(x)g'(x)(y-x)^2 \right| \\
&\qquad + \frac{1}{2}\left| f''(x)g(y) - f''(x)g(x) \right| (y-x)^2 \\
&= |g(y)| \left| f(y) - f(x) - f'(x)(y-x) - \frac{1}{2}f''(x)(y-x)^2 \right| \\
&\qquad + |f(x)| \left| g(y) - g(x) - g'(x)(y-x) - \frac{1}{2}g''(x)(y-x)^2 \right| \\
&\qquad + |f'(x)|\,|y-x|\,|g(y) - g(x) - g'(x)(y-x)| \\
&\qquad + \frac{1}{2}|f''(x)|\,|g(y) - g(x)|\,(y-x)^2 \\
&< \frac{|g(y)|}{4(|g(x_0)|+1)}\epsilon(y-x)^2 + \frac{|f(x)|}{4(|f(x_0)|+1)}\epsilon(y-x)^2 \\
&\qquad + \frac{|f'(x)|}{4(|f'(x_0)|+1)}\epsilon(y-x)^2 + \frac{|f''(x)|}{4(|f''(x_0)|+1)}\epsilon(y-x)^2 \\
&< \epsilon(y-x)^2.
\end{aligned}$$

□

COROLLARY 1. *All polynomials are WLUD^2 on $\mathcal{N}$.*

PROOF. Using Proposition 8 and Proposition 9, it suffices to show that the function $f(x) = x$ is WLUD^2 on $\mathcal{N}$. But that follows readily from the fact that, for all $x, y \in \mathcal{N}$, we have that

$$\left| f(y) - f(x) - f'(x)(y-x) - \frac{1}{2}f''(x)(y-x)^2 \right| = |y - x - (y-x)| = 0.$$

□

PROPOSITION 10. *Let $A, B \subseteq \mathcal{N}$ be open and let $g : A \to B$ be WLUD^2 at $x_0 \in A$ and $f : B \to \mathcal{N}$ be WLUD^2 at $g(x_0)$. Then $f \circ g : A \to \mathcal{N}$ is WLUD^2 at x_0.*

PROOF. Let $\epsilon > 0$ in $\mathcal{N}$ be given. As g and g^2 are WLUD^2 at x_0, and f is WLUD^2 at $g(x_0)$, there exist $\delta_1, \delta_2, \delta_3 > 0$ in $\mathcal{N}$ such that

$$\left| f(y) - f(x) - f'(x)(y-x) - \frac{1}{2}f''(x)(y-x)^2 \right| < \frac{\epsilon}{3\left(1 + |g'(x_0)|\right)^2}(y-x)^2$$

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for every $x, y \in B \cap B(g(x_0), \delta_1)$;

$$\begin{aligned}&\left|g(y) - g(x) - g'(x)(y-x) - \frac{1}{2}g''(x)(y-x)^2\right| \\ &\quad < \frac{\epsilon}{3\Big(\left|f'\big(g(x_0)\big)\right| + \left|f''\big(g(x_0)\big)g(x_0)\right| + 2\Big)}(y-x)^2\end{aligned}$$

for every $x, y \in A \cap B(x_0, \delta_2)$; and

$$\begin{aligned}&\left|g^2(y) - g^2(x) - 2g(x)g'(x)(y-x) - \Big(g'(x)^2 + g(x)g''(x)\Big)(y-x)^2\right| \\ &\quad < \frac{\epsilon}{3\Big(\left|f''\big(g(x_0)\big)\right| + 1\Big)}(y-x)^2.\end{aligned}$$

for every $x, y \in A \cap B(x_0, \delta_3)$.

Moreover, as f is C^2 at $g(x_0)$ and g is C^2 at x_0, there exists a $\delta_4 > 0$ in $\mathcal{N}$ such that for every $x, y \in A \cap B(x_0, \delta_4)$ we have that $|g(y) - g(x)| < \delta_1$, $|f'(g(x)) - f'(g(x_0))| < 1$, and $|f''(g(x))g(x) - f''(g(x_0))g(x_0)| < 1$. Finally, since g is WLUD at x_0, there exists $\delta_5 > 0$ in $\mathcal{N}$ such that for every $x, y \in A \cap B(x_0, \delta_5)$ we have that

$$|g(y) - g(x)| < (1 + |g'(x_0)|)\,|y - x|.$$

Let $\delta = \min\{\delta_1, \delta_2, \delta_3, \delta_4, \delta_5\}$. Then for every $x, y \in A \cap B(x_0, \delta)$ we have that

$$\begin{aligned}&\left|f\big(g(y)\big) - f\big(g(x)\big) - g'(x)f'\big(g(x)\big)(y-x) - \frac{1}{2}\Big(g''(x)f'\big(g(x)\big) + g'(x)^2 f''\big(g(x)\big)\Big)(y-x)^2\right| \\
&\le \left|f\big(g(y)\big) - f\big(g(x)\big) - f'\big(g(x)\big)\Big(g(y) - g(x)\Big) - \frac{1}{2}f''\big(g(x)\big)\Big(g(y) - g(x)\Big)^2\right| \\
&\quad + \left|f'\big(g(x)\big) - f''\big(g(x)\big)g(x)\right| \left|g(y) - g(x) - g'(x)(y-x) - \frac{1}{2}g''(x)(y-x)^2\right| \\
&\quad + \frac{1}{2}\left|f''\big(g(x)\big)\right| \left|g(y)^2 - g(x)^2 - 2g(x)g'(x)(y-x) - \Big(g'(x)^2 + g(x)g''(x)\Big)(y-x)^2\right| \\
&< \frac{\epsilon}{3\left(1 + |g'(x_0)|\right)^2}\left(g(y) - g(x)\right)^2 + \frac{\left|f'\big(g(x)\big) - f''\big(g(x)\big)g(x)\right|}{3\Big(\left|f'\big(g(x_0)\big)\right| + \left|f''\big(g(x_0)\big)g(x_0)\right| + 2\Big)}\epsilon(y-x)^2 \\
&\quad + \frac{\left|f''\big(g(x)\big)\right|}{3\Big(\left|f''\big(g(x_0)\big)\right| + 1\Big)}\epsilon(y-x)^2 \\
&< \epsilon(y-x)^2.\end{aligned}$$

□

So far we have shown that the LUD class of functions is contained in the WLUD class of functions, which in turn is contained in the C^1 class of functions. To complete our discussion of the basic properties of WLUD functions, we show that these inclusions are strict, meaning that LUD, WLUD, and C^1 are indeed distinct concepts in the non-Archimedean case while they are equivalent in the classical case.

NOTATION 2. For the next two examples we will use the following notation for convenience: Given $x, y \in \mathcal{R} \setminus \{0\}$, define $z = x - x[\lambda(x)]d^{\lambda(x)}$, $w = y - y[\lambda(y)]d^{\lambda(y)}$, and let $\lambda(x) = l/m$ and $\lambda(y) = j/k$ in lowest terms.

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We begin with a function that is C^1 and WLUD but is not LUD.

EXAMPLE 1. Let $f : (-1, 1) \to \mathcal{R}$ be given by

$$f(x) = \begin{cases} xd^m & m > 2\lambda(x),\ z \geq d^{2\lambda(x)}, \text{ and } \lambda\left(z - d^{2\lambda(x)}\right) \leq m \\ 0 & \text{else.} \end{cases}$$

We note that for any $r \in \mathbb{R}$ and $q \in \mathbb{Q}$, the set $\{x \in (-1, 1) \mid \lambda(x) = q, x[q] = r, z \geq d^{2q}, \lambda\left(z - d^{2q}\right) \leq m\}$ is clopen in $\mathcal{R}$, and thus f is locally linear everywhere but zero. Hence we have that f is differentiable for $x \neq 0$ with

$$f'(x) = \begin{cases} d^m & m > 2\lambda(x),\ z \geq d^{2\lambda(x)}, \text{ and } \lambda\left(z - d^{2\lambda(x)}\right) \leq m \\ 0 & \text{else,} \end{cases}$$

which is similarly continuous. Moreover, for $x \neq 0$, we have

$$\begin{aligned} \frac{f(x) - f(0)}{x} = \frac{f(x)}{x} &= \begin{cases} d^m & m > 2\lambda(x),\ z \geq d^{2\lambda(x)}, \text{ and } \lambda\left(z - d^{2\lambda(x)}\right) \leq m \\ 0 & \text{else} \end{cases} \\ &= f'(x) \ll d^{2\lambda(x)}. \end{aligned}$$

Therefore $\lim_{x\to 0} f(x)/x = \lim_{x\to 0} f'(x) = 0 = f'(0)$ and hence f is C^1.

Moreover, as we will show, f is WLUD as well. For $x_0 \neq 0$, this is trivial, as f is locally linear. For $x_0 = 0$, let $\epsilon > 0$ be given, choose $\delta = d^2\epsilon$, and let $x, y \in (-\delta, \delta)$ be given. We have 3 cases.

Case 1. *If $f(x) = 0$, $f(y) \neq 0$, then we must have $d^k \ll |y - x|$, and so*

$$|f(y) - f(x) - f'(x)(y - x)| = |yd^k| \ll \delta|y - x| \ll \epsilon|y - x|.$$

Case 2. *If $f(y) = 0$, $f(x) \neq 0$, then we must have $d^m \ll |y - x|$, and so*

$$|f(y) - f(x) - f'(x)(y - x)| = |xd^m + d^m(y - x)| = |y|d^m \ll \delta|y - x| \ll \epsilon|y - x|.$$

Case 3. *If $f(x), f(y) \neq 0$, then*

$$\begin{aligned} |f(y) - f(x) - f'(x)(y - x)| = |y||d^k - d^m| &\ll \delta d^{-1}\left|d^{2\lambda(y)} - d^{2\lambda(x)}\right| \\ &\ll \delta d^{-1}|y - x| \ll \epsilon|y - x|. \end{aligned}$$

We will now show f is not LUD at 0. Let a basic open neighbourhood $(-d^n, d^n)$ of 0 be given. Choose $\epsilon = d^{2n}$ and let $\delta > 0$ be given. Let $N = \max\{\lambda(\delta), 4n\}$, $q = (2nN - 1)/N$, $x = d^q + d^{2q}$, and $y = x + d^m$. Then $x, y \in U$, $m = N \geq 4n > 2n$, and

$$|f(y) - f(x) - f'(x)(y - x)| = yd^m = y|y - x| > \epsilon|y - x|.$$

Thus, f is C^1 but not LUD. In fact, f', being locally constant away from zero with $\lim_{x\to 0} f'(x) = 0$, is differentiable with derivative zero everywhere, and so f is C^∞ but not LUD.

This shows that LUD is a strictly stronger condition than WLUD. In the next example, we will show that WLUD is a strictly stronger condition than C^1.

EXAMPLE 2. Let $f : (-1, 1) \to \mathcal{R}$ be given by

$$f(x) = \begin{cases} z^2 & z < d^{2\lambda(x)} \text{ or } \left|z - d^{2\lambda(x)}\right| \ll d^m \\ z^2 + 2xz & z > d^{2\lambda(x)} \text{ and } \lambda\left(z - d^{2\lambda(x)}\right) \leq m \\ 0 & x = 0. \end{cases}$$

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First we will show that f is C^1. Just as in the previous example, the pieces on which f is defined (other than $\{0\}$ of course) are clopen sets. Thus, for $x_0 \neq 0$, f is locally a polynomial, and we have that f is C^1 on $(-1,1) \setminus \{0\}$ with

$$f'(x) = \begin{cases} 2z & z < d^{2\lambda(x)} \text{ or } \left|z - d^{2\lambda(x)}\right| \ll d^m \\ 4z + 2x & z > d^{2\lambda(x)} \text{ and } \lambda\left(z - d^{2\lambda(x)}\right) \leq d^m. \end{cases}$$

For $x_0 = 0$, we have

$$\begin{aligned} \left|\frac{f(x)}{x}\right| &= \begin{cases} \left|\frac{z}{x} z\right| & z < d^{2\lambda(x)} \text{ or } \left|z - d^{2\lambda(x)}\right| \ll d^m \\ \left|\frac{z}{x}(z + 2x)\right| & z > d^{2\lambda(x)} \text{ and } \lambda\left(z - d^{2\lambda(x)}\right) \leq d^m \end{cases} \\ &\leq 3|x|, \end{aligned}$$

and hence $f'(0) = \lim_{x \to 0} f(x)/x = 0$. Moreover, by our expression for f', we have $\lim_{x \to 0} f'(x) = 0$, and so f is C^1 on $(-1,1)$.

Now we will show that f is not WLUD at 0. Let $\epsilon = d$, let $\delta > 0$ be given, let $n \in \mathbb{N}$ be such that $n \geq 2$ and $d^n < \delta$, let $m = 4n$, and let $l = 4n^2 + 1$. Then l and m are co-prime and

$$\frac{m}{3} = \frac{4}{3}n > n + \frac{1}{4n} = \frac{4n^2+1}{4n} = \frac{l}{m} > n.$$

Let $x = d^{l/m} + d^{2l/m}$, $y = d^{l/m} + d^{2l/m} + d^m$. Then, $|x|$, $|y| < 2d^{l/m} \ll d^n < \delta$, and so $x, y \in B(0, \delta)$. Moreover,

$$f(y) - f'(x)y = w^2 + 2yw - 2zy \geq 2y(w - z) = 2yd^m \geq 0,$$

$$f'(x)x - f(x) = 2zx - z^2 = 2z(x - z) = 2d^{3l/m} \geq 0,$$

and so,

$$\begin{aligned} |f(y) - f(x) - f'(x)(y - x)| &= f(y) - f'(x)y + f'(x)x - f(x) \\ &\geq f'(x)x - f(x) = 2d^{3l/m} \gg d^m \gg \epsilon|y - x|. \end{aligned}$$

3. Calculus Theorems

The following lemma is a trivial consequence of the fact that a WLUD function is C^1.

LEMMA 2. *Let $A \subseteq \mathcal{N}$ be open and let $f : A \to \mathcal{N}$ be WLUD at $x_0 \in A$. Then for every $\epsilon > 0$ in $\mathcal{N}$, there exists $\delta > 0$ in $\mathcal{N}$ such that $B(x_0, \delta) \subset A$ and, for every $x, y \in B(x_0, \delta)$, we have that*

$$|f(y) - f(x) - f'(x_0)(y - x)| < \epsilon\,|y - x|\,.$$

REMARK 4. *As the preceding lemma will be the primary tool we will use to prove our major calculus theorems with WLUD, it is noteworthy to point out that the converse of the preceding lemma is also true for differentiable functions.*

PROPOSITION 11. *Let $A \subseteq \mathcal{N}$ be open, let $x_0 \in A \subset \mathcal{N}$ and let $f : A \to \mathcal{N}$ be a differentiable function such that for every $\epsilon > 0$ in $\mathcal{N}$, there exists $\delta > 0$ in $\mathcal{N}$ such that, for every $x, y \in B(x_0, \delta) \cap A$, we have that $|f(y) - f(x) - f'(x_0)(y - x)| < \epsilon\,|y - x|$. Then f is WLUD at x_0.*

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PROOF. It suffices to prove that f' is C^1 at x_0. Let $\epsilon > 0$ in $\mathcal{N}$ be given. Then there exists $\delta > 0$ in $\mathcal{N}$ such that $B(x_0, \delta) \subset A$ and, for every $x, y \in B(x_0, \delta)$, we have

$$|f(y) - f(x) - f'(x_0)(y - x)| < \frac{1}{2}\epsilon|y - x|.$$

Let $x \in B(x_0, \delta)$ be given. Then as f is differentiable at x, there exists $y \in B(x_0, \delta)$ such that

$$\left|\frac{f(y) - f(x)}{y - x} - f'(x)\right| < \frac{1}{2}\epsilon,$$

and so,

$$|f'(x) - f'(x_0)| \le \left|f'(x) - \frac{f(y) - f(x)}{y - x}\right| + \left|\frac{f(y) - f(x)}{y - x} - f'(x_0)\right| < \epsilon.$$

□

The following lemma, which is unrelated to the properties of WLUD functions, was proved in [**16**] (Lemma 4.1) .

LEMMA 3. *Let $\delta_1 > 0$ in $\mathcal{N}$ be given and let $\phi : B(0, \delta_1) \to \mathcal{N}$ be such that $|\phi(t)| \le c|t|$ for every $t \in B(0, \delta_1)$, where $0 < c \ll 1$. For $m \in \mathbb{N}$ let $\phi^{[m]} = \underbrace{\phi \circ \cdots \circ \phi}_{m \text{ times}}$ and set $\phi^{[0]}$ to be the identity map. Let $\delta \in \mathcal{N}$ be such that $0 < \delta \le (1 - c)\delta_1$ and let $\psi(t) = \sum_{m=0}^{\infty} \phi^{[m]}(t)$, for every $t \in B(0, \delta)$. Then*

(i) *$|\psi(t)| \le \frac{|t|}{1-c}$; and*
(ii) *$\psi(t) - \phi(\psi(t)) = t$.*

LEMMA 4. *Let $A \subseteq \mathcal{N}$ be open and let $f : A \to \mathcal{N}$ be WLUD on A with $f'(x_0) \neq 0$ for some $x_0 \in A$ and with $f(x_0) = y_0$. Then there exist $\delta, \eta > 0$ in $\mathcal{N}$ and a function F defined on $B(y_0, \eta)$ such that*

(i) *$B(x_0, \delta) \subseteq A$;*
(ii) *$f|_{B(x_0,\delta)}$ is injective;*
(iii) *$B(y_0, \eta) \subseteq f(B(x_0, \delta))$ and $F(B(y_0, \eta) \subseteq B(x_0, \delta)$;*
(iv) *$f(F(x)) = x$ for every $x \in B(y_0, \eta)$; and*
(v) *F is WLUD on $B(y_0, \eta)$ with $F' = 1/f' \circ F$.*

PROOF. Without loss of generality, we may assume that $x_0 = 0$ and $y_0 = 0$, for if this is not the case, then we can replace $f(x)$ with $\tilde{f}(x) = f(x + x_0) - y_0$. Moreover, without loss of generality, we may assume $f'(x_0) > 0$, for if $f'(x_0) < 0$ we could apply this proof to $(-f)$ and get the desired result.

By Proposition 1, f is C^1, and so there exists $\omega > 0$ in $\mathcal{N}$ such that $f'(x) \ge \frac{1}{2}f'(0) > 0$ for every $x \in B(x_0, \omega) \cap A$. Let $L = f'(0)$. Let $\phi(x) = x - \frac{1}{L}f(x)$. It follows that $\phi'(x) = 1 - \frac{1}{L}f'(x)$ and so $\phi'(0) = 0$. Let $c \in \mathcal{N}$ be such that $0 < c \ll 1$. As ϕ is WLUD at 0, then by Lemma 2, there exists $\delta_0 > 0$ in $\mathcal{N}$ such that $B(0, \delta_0) \subseteq A$ and, for every $s, t \in B(0, \delta_0)$, we have that

$$|\phi(s) - \phi(t) - \phi'(0)(s - t)| < c\,|s - t|\,.$$

Thus,

$$|\phi(s) - \phi(t)| < c\,|s - t|\,. \tag{3.1}$$

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Let $s, t \in B(0, \delta_0)$ be such that $f(s) = f(t)$. Then

$$|\phi(s) - \phi(t)| = |s - t| \leq c\,|s - t|\,.$$

As $c \ll 1$, it follows that $s = t$, and thus $f|_{B(0,\delta_0)}$ is injective. By Lemma 2, there exists $\delta_f > 0$ in $\mathcal{N}$ such that for every $s, t \in B(0, \delta_f)$ we have that

$$|f(s) - f(t) - L(s - t)| < \frac{L}{2}\,|s - t|\,.$$

Let $\delta = \min\{(1 - c)\delta_0, \omega, \delta_f\}$. Then $B(0, \delta) \subset B(0, \delta_0) \subset A$ and thus $f|_{B(0,\delta)}$ is injective. This shows (i) and (ii).

By Equation (3.1) with $t = 0$, we have that $|\phi(s)| < c\,|s|$ for every $s \in B(0, \delta)$, and so we have a function ψ with properties of that in Lemma 3. Let $\eta = L(1-c)\delta$ and define $F(x) = \psi(\frac{x}{L})$ for every $x \in B(0, \eta)$. Thus for every $x \in B(0, \eta)$ we have that

$$|F(x)| = \left|\psi\left(\frac{x}{L}\right)\right| \leq \frac{|x|}{L(1-c)} < \frac{\eta}{L(1-c)} = \delta.$$

Thus $F(B(0, \eta)) \subseteq B(0, \delta)$. Furthermore, for every $x \in B(0, \delta)$, we have that

$$x - \phi(x) = \frac{f(x)}{L}.$$

Let $x \in B(0, \eta)$. Then

$$\frac{|x|}{L} < (1 - c)\delta < \delta.$$

Thus $\frac{x}{L} \in B(0, \delta)$ and hence

$$\frac{x}{L} - \phi\left(\frac{x}{L}\right) = \frac{1}{L} f\left(\frac{x}{L}\right).$$

Moreover, we have by Lemma 3 that

$$\psi\left(\frac{x}{L}\right) - \phi\left(\psi\left(\frac{x}{L}\right)\right) = \frac{x}{L}$$

and thus

$$\frac{1}{L} f\left(\psi\left(\frac{x}{L}\right)\right) = \frac{x}{L}.$$

It follows that for every $x \in B(0, \eta)$,

$$f(F(x)) = f\left(\psi\left(\frac{x}{L}\right)\right) = x$$

and hence $B(0, \eta) \subseteq f(B(0, \delta))$, as $F(x) \in B(0, \delta)$ for every $x \in B(0, \eta)$. This shows (iii) and (iv).

Now, for any $x \in B(0, \delta)$ we have

$$|f(s) - f(t) - L(s - t)| < \frac{L}{2}\,|s - t|$$

and thus

$$|f(s) - f(t)| \geq L\,|s - t| - \frac{L}{2}\,|s - t| = \frac{L}{2}\,|s - t|\,.$$

Let $\epsilon > 0$ be given. As f is WLUD at 0, there exists $\delta_2 > 0$ such that for every $x, y \in B(0, \delta_2)$ we have

$$|f(y) - f(x) - f'(x)(y - x)| < \frac{L^2}{4}\epsilon|y - x|.$$

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Let $\delta_3 = \min\{L\delta_2/2, \eta, \delta\}$, let $x, y \in B(0, \delta_3)$ be given, and let $t_x = F(x)$ and $t_y = F(y)$. Then

$$|t_y - t_x| \leq \frac{2}{L}|f(t_y) - f(t_x)| = \frac{2}{L}|y - x| < \delta_2$$

Thus,

$$\begin{aligned}\left|F(y) - F(x) - \frac{1}{f'(F(x))}(y-x)\right| &= \left|\frac{1}{f'(F(x))}\right| |y - x - f'(F(x))(F(y) - F(x))| \\ &\leq \frac{2}{L}|f(t_y) - f(t_x) - f'(t_x)(t_y - t_x)| \\ &< \epsilon\frac{L}{2}|t_y - t_x| \leq \epsilon|y - x|,\end{aligned}$$

which shows (v). □

THEOREM 3.1. *Let $A \subseteq \mathcal{N}$ be open and let $f : A \to \mathcal{N}$ be WLUD on A with $f'(x_0) \neq 0$ for some $x_0 \in A$. Then there exists a neighborhood U of x_0 in A such that*

(i) *$f|_U$ is injective;*
(ii) *$f(U)$ is open; and*
(iii) *f^{-1} exists and is WLUD on $f(U)$ with $(f^{-1})' = 1/f' \circ f^{-1}$.*

PROOF. By Lemma 4, there exists a neighborhood U_0 of x_0 in A such that f is injective on U_0. As f is C^1 and $f'(x_0) \neq 0$, there exists a neighborhood U_1 of x_0 in A such that $f'(x) \neq 0$ for every $x \in U_1$. Let $U = U_0 \cap U_1$. Then U is a neighborhood of x_0 and $f|_U$ is injective.

Let $x \in U$ and let $y = f(x)$. Lemma 4 applied to $f|_U$ at x gives a δ, η, and F as stated in the lemma, for which $B(y, \eta) \subseteq f(B(x, \delta)) \subseteq f(U)$. As this holds for every $x \in U$, we have that $f(U)$ is open.

As f is injective f^{-1} exists on $f(U)$ and for any $y \in f(U)$,

$$f(f^{-1}(y)) = y = f(F(y))$$

and so we have that $F = f^{-1}$, which is WLUD at y with $(f^{-1})' = 1/f' \circ f^{-1}$ by lemma 4.

□

THEOREM 3.2 (local intermediate value theeorem). *Let $A \subseteq \mathcal{N}$ be open, let $f : A \to \mathcal{N}$ be WLUD on A and let $x_0 \in A$ be such that $f'(x_0) \neq 0$. Then there exists a neighborhood U of x_0 in A such that f has the intermediate value property on U. That is, for every $a, b \in U$ with $a < b$, if c is between $f(a)$ and $f(b)$, then there exists $x \in (a, b)$ such that $f(x) = c$.*

PROOF. Without loss of generality, we may assume $f'(x_0) > 0$. By Lemma 2, there exists $\delta > 0$ in $\mathcal{N}$ such that $B(x_0, \delta) \subset A$ and, for all $x \neq y$ in $B(x_0, \delta)$, we have that

$$|f(y) - f(x) - f'(x_0)(y - x)| < \frac{f'(x_0)}{2}|y - x|$$

and thus

$$\frac{f(y) - f(x)}{y - x} > f'(x_0) - \frac{f'(x_0)}{2} = \frac{f'(x_0)}{2} > 0.$$

Hence f is strictly increasing on $B(x_0, \delta)$. Applying Theorem 3.1 to f gives a neighborhood $U_0 \subseteq B(x_0, \delta)$ of x_0 such that $f(U_0)$ is open. Let $\epsilon > 0$ in $\mathcal{N}$ be

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such that $B(f(x_0), \epsilon) \subseteq f(U_0)$ and let $U = f^{-1}(B(f(x_0), \epsilon))$, which is an open neighborhood of x_0. Let $a, b \in U$ be such that $a < b$ and let $c \in (f(a), f(b))$ be given. As $f(a), f(b) \in B(f(x_0), \epsilon)$ and $B(f(x_0), \epsilon)$ is a convex set, we have that $c \in B(f(x_0), \epsilon)$. Thus there exists $x \in U = f^{-1}(B(f(x_0), \epsilon))$ such that $f(x) = c$. As f is strictly increasing on U, it follows that $x \in (a, b)$. □

THEOREM 3.3 (local mean value theorem). *Let $A \subseteq \mathcal{N}$ be open, let $f : A \to \mathcal{N}$ be WLUD^2 at $x_0 \in A$ and assume that $f''(x_0) \neq 0$. Then there exists a neighborhood U of x_0 in A such that f has the mean value property on U. That is, for every $a, b \in U$ with $a < b$, there exists $c \in (a, b)$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

PROOF. We may assume without loss of generality that $f''(x_0) > 0$. By Proposition 6, f'' is continuous at x_0. Thus there exists $\delta_1 > 0$ such that $U_1 = B(x_0, \delta_1) \subset A$ and, for every $x \in U_1$ we have that

$$|f''(x) - f''(x_0)| < \frac{1}{4} f''(x_0).$$

As f is WLUD^2 at x_0, there exists $\delta_2 > 0$ in $\mathcal{N}$ such that, for every $x, y \in U_1$ with $0 < |y - x| < \delta_2$, we have that

$$\left| f(y) - f(x) - f'(x)(y - x) - \frac{1}{2} f''(x)(y - x)^2 \right| < \frac{1}{4} f''(x_0)(y - x)^2.$$

Let $\delta = \min\{\delta_1, \delta_2\}$. Then it follows that, for every $x, y \in B(x_0, \delta)$, we have that

$$\begin{aligned} f(y) - f(x) - f'(x)(y - x) &> \left(\frac{1}{2} f''(x) - \frac{1}{4} f''(x_0) \right) (y - x)^2 \\ &> \frac{1}{8} f''(x_0)(y - x)^2 > 0. \end{aligned} \tag{3.2}$$

Applying Theorem 3.2 to f' at x_0 gives a neighborhood U_0 of x_0 in A such that f' has the intermediate value property in U_0. Let $U = U_0 \cap B(x_0, \delta)$ and let $a, b \in U$ with $a < b$ be given. By Equation (3.2) we have that $f(b) > f(a) + f'(a)(b - a)$, and thus

$$f'(a) < \frac{f(b) - f(a)}{b - a}.$$

Similarly, we have that $f(a) > f(b) + f'(b)(a - b)$, and thus

$$f'(b) > \frac{f(b) - f(a)}{b - a}$$

Thus, by Theorem 3.2, there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

□

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