PHYS 2490  
Final Exam  
Wednesday, April 17, 2013  
1:30- 4:30 PM

**Instructions:** Please read the following instructions before you start working on your problems.

- Write your name and student number on each of the provided examination booklets.

- All course materials must remain closed during the exam. But you are allowed to have one eight and a half by eleven inch sheet of notes besides the formula pages provided with the exam.

- **Write all necessary steps to get full credit**

- The exam is two parts (Total 55 marks):

  - **Part A:** Work on ANY FOUR problems (out of five) for a total of 40 marks;
  - **Part B:** Work on ONLY ONE problem (out of two) for 15 marks.

**Good Luck!**
Part A (40 marks): Work on ANY FOUR of the following five problems.

Problem 1 (10 marks): Let \( f(x) \) be given by
\[
f(x) = x^2 \text{ for } -1 \leq x \leq 1.
\]
(a) Find the Fourier series of period 2 that represents \( f(x) \).
(b) Use Parseval’s Theorem, applied to \( f(x) \) and the Fourier series found in part (a), to find the exact value of the infinite sum
\[
\sum_{n=1}^{\infty} \frac{1}{n^4}.
\]

Problem 2 (10 marks): If a 10 kg block of rock salt is placed in water, it dissolves at a rate proportional to the amount of salt still undissolved. If 2 kg dissolve during the first 10 minutes, how long will it be until only 2 kg remain undissolved?

Problem 3 (10 marks): Find the general solution of the following second order differential equation:
\[
\frac{d^2y}{dx^2} - 5 \frac{dy}{dx} + 6y = e^{2x} + 10 \cos x + 3x.
\]

Problem 4 (10 marks): Solve the following second order differential equation using the generalized power series method (Frobenius method).
\[
x(1-x) \frac{d^2y}{dx^2} - 2 \frac{dy}{dx} + 2y = 0.
\]

Continued on the next page
Problem 5 (10 marks): Let

\[ f(x) = \begin{cases} 
0 & \text{if } -1 \leq x < 0 \\
2 & \text{if } 0 < x \leq 1 
\end{cases} \]

and consider the Legendre series for \( f(x) \):

\[ f(x) = \sum_{l=0}^{\infty} c_l P_l(x) \]

a) Show that \( c_l = 0 \) for even \( l > 0 \); i.e. for \( l = 2, 4, 6, \ldots \). **Hint:** Write \( f(x) = (\text{constant}) + g(x) \), where \( g(x) \) is an odd function on the interval \([-1, 1]\).

b) Compute the first three non-zero coefficients in the expansion: \( c_0, c_1, c_3 \). (Note: the \( P_l(x) \) are given on the formula sheet.)

c) Use the result of part a) to show that

\[ \int_{0}^{1} P_l(x) \, dx = 0 \text{ for even } l = 2, 4, 6, \ldots \]

Part B of the exam is on the next page
Part B (15 marks): Work on **ONLY ONE** of the following two problems.

**Problem 6 (15 marks):** Find the steady-state temperature distribution for the semi-infinite plate problem if the temperature on the bottom edge is $T(x, 0) = 100^\circ$, the temperature of the other sides is $0^\circ$, and the width of the plate is 10 cm.

\[ \nabla^2 T(x, y) = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0. \]

**Problem 7 (15 marks):** Find the steady-state temperature distribution in a solid cylinder of height 10 and radius 1 if the top and curved surface of the cylinder are held at $0^\circ$ and the base is held at $100^\circ$.

**Hint:** Laplace’s Equation in cylindrical coordinates is:

\[ \nabla^2 u = \frac{1}{r \frac{\partial}{\partial r}} \left( r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = 0; \]

but because of the symmetry of this problem with respect to the z-axis (the axis of the cylinder), $u$ is independent of $\theta$. Thus, $u = u(r, z)$; and Laplace’s Equation becomes:

\[ \nabla^2 u = \frac{1}{r \frac{\partial}{\partial r}} \left( r \frac{\partial u}{\partial r} \right) + \frac{\partial^2 u}{\partial z^2} = 0. \]
Formulae for PHYS2490

1. Average value of a function

\[ <f> = \frac{1}{b-a} \int_a^b f(x)dx \]

2. Real and Complex Fourier Series of period 2L

\[ f(x) = f(x + 2L) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right) \]

\[ f(x) = \sum_{-\infty}^{\infty} c_n e^{i\frac{nx\pi}{L}} \]

3. Coefficients

\[ a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right)dx \]

\[ b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right)dx \]

\[ c_n = \frac{1}{2L} \int_{-L}^{L} f(x) e^{-i\frac{nx\pi}{L}}dx \]

4. Parseval's Theorem

\[ <f^2> = \left(\frac{a_0}{2}\right)^2 + \frac{1}{2} \sum_{n=1}^{\infty} a_n^2 + \frac{1}{2} \sum_{n=1}^{\infty} b_n^2 = \sum_{-\infty}^{\infty} |c_n|^2 \]

5. First Order Linear Differential Equation

\[ y = e^{-t} \int Qe^t dx + ce^{-t} \]

\[ I = \int P dx \]

6. Laplace Transform

\[ L(y) = \int_0^{\infty} y(t)e^{-pt}dt = Y(p) \]

\[ L(y') = pY - y_0 \]

\[ L(y'') = p^2Y - py_0 - y'_0 \]

7. Fourier Transform

\[ f(x) = \int_{-\infty}^{\infty} g(\alpha)e^{i\alpha x}d\alpha \]

\[ g(\alpha) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x)e^{-i\alpha x}dx \]
Formulae for PHYS2490

Curvilinear Coordinates:

(a) Arc length
\[ ds^2 = \sum_{i=1}^{3} h_i^2 dx_i^2 \]

(b) Gradient
\[ \vec{\nabla}u = \sum_{i=1}^{3} \frac{\partial}{h_i \partial x_i} \]

(c) Laplacian
\[ \nabla^2 u = \frac{1}{h_1 h_2 h_3} \left[ \frac{\partial}{\partial x_1} \left( \frac{h_2 h_3}{h_1} \frac{\partial u}{\partial x_1} \right) + \frac{\partial}{\partial x_2} \left( \frac{h_1 h_3}{h_2} \frac{\partial u}{\partial x_2} \right) + \frac{\partial}{\partial x_3} \left( \frac{h_1 h_2}{h_3} \frac{\partial u}{\partial x_3} \right) \right] \]

(d) Cartesian Coordinates \((x, y, z)\)
\[ h_1 = h_2 = h_3 = 1 \]

(e) Cylindrical Coordinates \((r, \theta, z)\)
\[ h_1 = 1, h_2 = r, h_3 = 1 \]

(f) Spherical Coordinates \((r, \theta, \phi)\)
\[ h_1 = 1, h_2 = r, h_3 = r \sin \theta \]

Gamma Function

(a) Gamma Function
\[ \Gamma(p) = \int_0^\infty x^{p-1}e^{-x}dx \quad p > 0 \]

(b) Factorial Function
\[ \Gamma(n+1) = n! \]
\[ \Gamma(p+1) = p\Gamma(p) \]
\[ \Gamma(1/2) = \sqrt{\pi} \]
Series Solutions of ODES

(a) Frobenius Method

\[ y(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \]

(b) Legendre's Equation

\[ (1 - x^2)y'' - 2xy' + l(l+1)y = 0 \]

(c) With \( x = \cos \theta \), this becomes

\[ \frac{1}{\sin \theta} \frac{d}{d\theta} (\sin \theta \frac{dy}{d\theta}) + l(l+1)y = 0 \]

(d) Legendre Polynomials

\[ P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2}(3x^2 - 1), \quad P_3(x) = \frac{1}{2}(5x^3 - 3x), \]

\[ P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3), \quad P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x) \]

(e) Orthogonality of Legendre Polynomials

\[ \int_{-1}^{1} P_l(x) P_m(x) dx = 0 \quad m \neq l \]

\[ \int_{-1}^{1} [P_l(x)]^2 dx = \frac{2}{2l+1}. \]

(f) Bessel's Equation

\[ x^2y'' + xy' + (x^2 - p^2)y = 0 \]

or if \( x = kr \)

\[ r^2y'' + ry' + (k^2r^2 - p^2)y = 0 \]

(g) Orthogonality of Bessel Functions

\[ \int_{0}^{1} x J_p(ax) J_p(bx) dx = 0 \quad if \ a \neq b \]

\[ \int_{0}^{1} x J_p(ax) J_p(ax) dx = \frac{1}{2} J^2_{p+1}(a) \quad if \ a = b \]

where \( a \) and \( b \) are the zeros of \( J_p(x) \) and also

\[ \int x^p J_{p-1}(x) dx = x^p J_p(x) \]
Partial Differential Equations

(a) Laplace's Equation
\[ \nabla^2 u = 0 \]

(b) Diffusion or Heat Flow
\[ \nabla^2 u = \frac{1}{\alpha^2} \frac{\partial u}{\partial t} \]

(c) Wave Equation
\[ \nabla^2 u = \frac{1}{v^2} \frac{\partial^2 u}{\partial t^2} \]
\[ f(x) = a_0 + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi}{\ell} x + b_n \sin \frac{n\pi}{\ell} x \right) \]

Here \( \ell = 1 \) and \( f(x) \) is an even function so \( b_n = 0 \) for all \( n \geq 1 \). Moreover,

\[ a_n = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \cos \frac{n\pi}{\ell} x \, dx = \frac{2}{\ell} \int_{0}^{\ell} f(x) \cos \frac{n\pi}{\ell} x \, dx \]

\[ = 2 \int_{0}^{1} x^2 \cos n\pi x \, dx, \quad n \geq 0. \]

\[ a_0 = 2 \int_{0}^{1} x^2 \, dx = \frac{2}{3} ; \quad \text{and, for } n \geq 1, \]

\[ a_n = 2 \int_{0}^{1} x^2 \cos n\pi x \, dx \]

\[ = 2 \left[ \frac{x^2}{n\pi} \sin n\pi x + \frac{2x}{n^2\pi^2} \cos n\pi x - \frac{2}{n^3\pi^3} \sin n\pi x \right]_{x=0}^{1} \]

\[ = \frac{4}{n^2\pi^2} \cos n\pi = \frac{4}{n^2\pi^2} (-1)^n \]

\[ \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2} = \left\{ \begin{array}{ll} \frac{2}{n^2\pi^2} & \text{if } n \text{ is even} \\ \frac{-2}{n^2\pi^2} & \text{if } n \text{ is odd} \end{array} \right. \]
Thus, \( f(x) = \frac{1}{3} - \frac{4}{\pi^2} \left[ \cos \pi x - \frac{1}{4} \cos 2\pi x + \frac{1}{9} \cos 3\pi x - \frac{1}{16} \cos 4\pi x + \cdots \right] \)

\[ = \frac{1}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos n\pi x \]

b) Applying Parseval's Theorem, we get:

\[ \frac{1}{2} \int_{-1}^{1} x^4 \, dx = \left( \frac{1}{3} \right)^2 + \frac{1}{2} \sum_{n=1}^{\infty} (a_n)^2 \quad [b_n = 0] \]

\[ \frac{1}{5} = \frac{1}{9} + \frac{1}{2} \sum_{n=1}^{\infty} \frac{16}{n^4 \pi^4} \]

\[ \frac{4}{45} = \frac{8}{\pi^4} \sum_{n=1}^{\infty} \frac{1}{n^4} \Rightarrow \]

\[ \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90} \]

2. \( M_0 = 10 \text{ kg} \)

\[ \frac{dM}{dt} = -\alpha M \quad (\alpha > 0) \]

\[ \frac{dM}{M} = -\alpha \, dt \]

\[ \ln(M(t)) = -\alpha t + a \]

At \( t = 0 \), \( M(t) = M_0 = 10 \) so \( a = \ln M_0 \)
Thus, \( \ln M(t) = -\alpha t + \ln M_0 \Rightarrow \)
\[
M(t) = M_0 e^{-\alpha t} = 10 e^{-\alpha t}
\]

At \( t = 10 \text{ min} \), \( M(t) = 10 e^{10\alpha} = 8 \Rightarrow \)
\[
e^{-10\alpha} = \frac{8}{10} \Rightarrow -10\alpha = \ln \left( \frac{8}{10} \right)
\]
\[
\Rightarrow \alpha = \frac{1}{10} \ln \left( \frac{10}{8} \right) = \frac{1}{10} \ln \left( \frac{5}{4} \right) \text{ min}^{-1}
\]

\( M(t) = 2 \text{ kg} \Rightarrow t = ? \)
\[
2 = 10 e^{-\alpha t} \Rightarrow -\alpha t = \ln \left( \frac{1}{5} \right)
\]
\[
\Rightarrow \alpha t = \ln 5 \Rightarrow t = \frac{1}{\alpha} \ln 5
\]
\[
= 10 \frac{\ln 5}{\ln(5/4)} = 7.2 \text{ min}
\]
3. \[ \frac{d^2y}{dx^2} - 5 \frac{dy}{dx} + 6y = e^{2x} + 10 \cos x + 3x \tag{1} \]

\[ \frac{d^2y}{dx^2} - 5 \frac{dy}{dx} + 6y = 0 \rightarrow y_c \]

\[(D^2 - 5D + 6) \ y = 0 \]
\[(D - 2)(D - 3) \ y = 0 \rightarrow y_c = c_1 e^{2x} + c_2 e^{3x} \]

\[ \frac{d^2y}{dx^2} - 5 \frac{dy}{dx} + 6y = e^{2x} \rightarrow y_p, \]

Let \[ y_p = A x e^{2x} \] (it is equal to one of the roots of the characteristic eqn.) Then

\[ y'_{p} = A (2x + 1) e^{2x} \text{ and} \]
\[ y''_{p} = A (4x + 4) e^{2x} \text{. Thus,} \]
\[ A e^{2x} \left[ 4x + 4 - 5 (2x + 1) + 6x \right] = 1 e^{2x} \]
\[ \rightarrow A = -1 \text{ and hence} \]
\[ y_{p} = -x e^{2x} \]

\[ \frac{d^2y}{dx^2} - 5 \frac{dy}{dx} + 6y = 10 \cos x \rightarrow y_{p_2} \]

First we write

\[ \frac{d^2y}{dx^2} - 5 \frac{dy}{dx} + 6y = 10 e^{ix} \rightarrow y_{p} \]
Then \( y_p = \Re (Y_p) \). Let \( Y_p = Be^{ix} \). Then,

\[
y_p' = iBe^{ix} \text{ and } y_p'' = -Be^{ix}. \text{ Thus,}
\]

\[
Be^{ix} \left[ -1 - 5i + 6 \right] = 10e^{ix}
\]

So \( 5B(1-i) = 10 \) \( \Rightarrow B = \frac{2}{1-i} = 1+i \). Hence,

\[
y_p = (1+i)e^{ix} = (1+i)(\cos x + i\sin x)
\]

\[
= \cos x - \sin x + i(\cos x + \sin x) \quad \text{Thus,}
\]

\[
y_{p_2} = \Re (Y_p) = \cos x - \sin x
\]

\[
\frac{d^2y}{dx^2} - 5 \frac{dy}{dx} + 6y = 3x \rightarrow y_{p_3}
\]

Let \( y_{p_3} = Cx + D \). Then

\[
y_{p_3}' = C \text{ and } y_{p_3}'' = 0. \text{ Thus,}
\]

\[
0 - 5C + 6(Cx + D) = 3x.
\]

\[
\begin{cases}
6C = 3 \rightarrow C = \frac{1}{2} \\
6D - 5C = 0 \rightarrow D = \frac{5}{6}C = \frac{5}{12}
\end{cases}
\]
Hence
\[ y_p^3 = \frac{1}{2} x + \frac{5}{12} = \frac{1}{12} (6x + 5) \]

 Altogether, the general solution of (x) is:
\[ y = y_c + y_{p_1} + y_{p_2} + y_{p_3} \]
\[ = c_1 e^{2x} + c_2 e^{3x} - x e^{2x} + \cos x - \sin x + \frac{1}{12} (6x + 5) \]

4. \( x (1-x) \frac{d^2 y}{dx^2} - 2 \frac{dy}{dx} + 2y = 0 \)

Let \( y = \sum_{n=0}^{\infty} a_n x^{n+s} \), \( a_0 \neq 0 \). Then
\[
(\lambda - x^2) \sum_{n=0}^{\infty} (n+s)(n+s-1) a_n x^{n+s-2}
- 2 \sum_{n=0}^{\infty} a_n (n+s)x^{n+s-1} + 2 \sum_{n=0}^{\infty} a_n x^{n+s} = 0
\]
\[
\sum_{n=0}^{\infty} (n+s)(n+s-1) a_n x^{n+s-1} - \sum_{n=0}^{\infty} (n+s)(n+s-1) a_n x^{n+s}
- \sum_{n=0}^{\infty} 2 a_n (n+s) x^{n+s-1} + \sum_{n=0}^{\infty} 2 a_n x^{n+s} = 0
\]
\[
\sum_{n=0}^{\infty} (n+s)(n+s-3) a_n x^{n+s-1} - \sum_{n=0}^{\infty} [(n+s)(n+s-1) - 2] a_n x^{n+s} = 0
\]
\[
\sum_{n=0}^{\infty} \frac{(n+1)(n-2)a_{n+1}}{(n+1)^2} x^n - \sum_{n=0}^{\infty} \frac{(n+1)(n+s-1)a_n}{(n+1)^2} x^n = 0
\]

\[
s(s-3)a_0 x^{s-1} + \sum_{n=0}^{\infty} \frac{(n+1)(n+s-2)a_{n+1} - (n+s)(n+s-1)a_n}{(n+1)^2} x^n = 0
\]

\[
s(s-3)a_0 = 0 \Rightarrow s(s-3) = 0 \quad (a_0 \neq 0)
\]

\[
\Rightarrow s = 0 \quad \text{or} \quad s = 3
\]

\[s = 0:\]
\[
\sum_{n=0}^{\infty} \frac{(n+1)(n-2)a_{n+1} - n(n-1)a_n}{(n+1)^2} x^n = 0
\]

\[
\Rightarrow a_{n+1} = \frac{n(n-1) - 2n}{(n+1)^2} a_n
\]

\[
\sum_{n=0}^{\infty} \frac{(n+1)(n-2)(a_{n+1} - a_n)}{x^n} = 0
\]

\[
(n+1)(n-2)(a_{n+1} - a_n) = 0 \quad \text{for all} \quad n \geq 0
\]

For \( n \neq 2 \), \((n+1)(n-2) \neq 0\) and hence \( a_{n+1} = a_n \) \( \text{for} \quad n \neq 2 \). Thus, \( a_1 = a_0 \) \( \text{and} \quad a_2 = a_1 \) \( (n=1) \).

Also, \( a_4 = a_3 \) \( (n=3) \), \( a_5 = a_4 \) \( (n=4) \), etc.
For $n = 2$, $(n+1)(n-2) = 0$, so $a_3 - a_2$ can be anything. This means $a_3$ is arbitrary. (like $a_6$).

$a_0 = a_1 = a_2$ and

\[ a_3 = a_4 = a_5 = \ldots \]

Thus,

\[ y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \ldots = a_0 + a_0 x + a_0 x^2 + a_3 x^3 + a_3 x^4 + a_3 x^5 + \ldots = a_0 \left( 1 + x + x^2 \right) + a_3 \left( x^3 + x^4 + x^5 + \ldots \right) = a_0 \left( 1 + x + x^2 \right) + a_3 \frac{x^3}{1-x} \]

Since this contains two arbitrary constants ($a_0$ and $a_3$), $y$ is the general solution of the second order linear ODE. So no need to proceed with $s = 3$.

Note: If you look for the solution corresponding to $s = 3$, $y = x^3 \sum_{n=0}^{\infty} b_n x^n$, you should get $b_{n+1} = b_n$ for all $n \geq 0$, so that

\[ y = b_0 x^3 (1 + x + x^2 + \ldots) = b_0 \frac{x^3}{1-x} \] (obtained above!)
5. \( f(x) = \begin{cases} 0 & \text{if } -1 \leq x < 0 \\ 2 & \text{if } 0 < x \leq 1 \end{cases} \)

a) \( f(x) = 1 + g(x) \), where

\[
g(x) = \begin{cases} -1 & \text{if } -1 \leq x < 0 \\ 1 & \text{if } 0 < x \leq 1 \end{cases}
\]

is an odd function on \([-1,1]\). Thus,

\[
g(x) = \sum_{\ell = 1}^{\infty} \frac{c_{\ell}}{\ell} \cos \ell \pi x \quad \text{and hence}
\]

\[
f(x) = 1 + \sum_{\ell = 1}^{\infty} \frac{c_{\ell}}{\ell} \cos \ell \pi x \quad \text{and hence}
\]

\[
a_2 = a_4 = \cdots = 0, \quad \sum_{0}^{\infty} c_0 = 1
\]

b) To compute the \( c_{\ell} \)'s, we could use either \( f \) or

\[
(\ell > 0) \quad \text{we already know} \quad c_0 = 1
\]

\[
c_{\ell} = \frac{2\ell+1}{2} \int_{-1}^{1} f(x) \cos \ell \pi x \, dx
\]

\[
= \frac{(2\ell+1)}{2} \int_{0}^{1} (2\ell+1) \cos \ell \pi x \, dx
\]

\[
= \begin{cases} 0 & \text{if } \ell \text{ is even} \\ \frac{(2\ell+1)}{2} \int_{0}^{1} \cos \ell \pi x \, dx & \text{if } \ell \text{ is odd} \end{cases}
\]

\[
\left\{ \begin{array}{l}
\text{valid for all } \ell \geq 1
\end{array} \right.
\]
Thus,
\[
c_1 = 3 \int_0^1 p_1(x) \, dx = 3 \int_0^1 x \, dx = \frac{3}{2}
\]
\[
c_3 = 7 \int_0^1 p_3(x) \, dx = \frac{7}{2} \int_0^1 (5x^2 - 3x) \, dx
\]
\[
= \frac{7}{2} \left[ \frac{5}{4} - \frac{3}{2} \right] = -\frac{7}{8} \quad \text{Hence}
\]
\[
l(x) = 1 \cdot p_0(x) + \frac{3}{2} \cdot p_1(x) - \frac{7}{8} \cdot p_3(x) + \ldots
\]

(\text{c}) \quad \text{from the parts (a) and (b) above,}
\[
c_l = (2l+1) \int_0^1 p_l(x) \, dx = 0 \quad \text{for} \quad l = 2, 4, 6, \ldots
\]

Thus, \( \int_0^1 p_l(x) \, dx = 0 \) \( \text{for} \quad l = 2, 4, 6, \ldots \)

Alternatively: \( \text{for} \quad l = 2, 4, 6, \ldots \):
\[
\int_0^1 p_l(x) \, dx = \frac{1}{2} \int_0^1 p_l(x) \, dx \quad (p_l \text{ is even})
\]
\[
= \frac{1}{2} \int_0^1 p_l(x) \, p_0(x) \, dx = 0 \quad \text{since}
\]
\[
p_l(x) \text{ and } p_0(x) \text{ are orthogonal on } [-1, 1].
\]
6. \[ \nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \] (x)

We look for solutions of the form:

\[ T(x, y) = X(x) Y(y), \] Substituting into (x),

we get

\[ Y \frac{d^2 X}{dx^2} + X \frac{d^2 Y}{dy^2} = 0 \quad \text{(* \( \frac{1}{xy} \))} \]

\[ \rightarrow \quad \frac{1}{x} \frac{d^2 X}{dx^2} + \frac{1}{y} \frac{d^2 Y}{dy^2} = 0 \]

or

\[ \frac{1}{x} \frac{d^2 X}{dx^2} = - \frac{1}{y} \frac{d^2 Y}{dy^2} ; \quad \text{the left-hand side is a function of } x \text{ only and the right-hand side a function of } y \text{ only.} \]

Since this has to hold for all \( x, y \), which can vary independently, this can happen if both sides are equal to the same constant, say \( \alpha \).

From the boundary condition

\[ T(x, y) \to 0 \quad \text{as} \quad y \to \infty \]

Thus, \( \alpha = -k^2 < 0 \). Hence
\[ \frac{1}{x} \frac{d^2 x}{dx^2} = - \frac{1}{y} \frac{d^2 y}{dy^2} = -\kappa^2 \]

\[ \begin{cases} \frac{d^2 x}{dx^2} + \kappa^2 x = 0 & \rightarrow x(x) = \left\{ \begin{array}{l} \cos \kappa x \\ \sin \kappa x \end{array} \right. \\
\frac{d^2 y}{dy^2} - \kappa^2 y = 0 & \rightarrow y(y) = \left\{ \begin{array}{l} e^{-\kappa y} \\ e^{\kappa y} \end{array} \right. \end{cases} \]

From \( T(10, y) = 0 \), we get \( x(0) = 0 \) so \( \cos \kappa x \) not acceptable!

From \( T(10, y) = 0 \), we get \( x(10) = \sin \kappa(10) = 0 \)

\[ 10 \kappa = n \pi \rightarrow \kappa = \frac{n\pi}{10} \quad \text{(Eigenvalue).} \]

So for each \( n = 1, 2, 3, \ldots \)

\[ \sin \frac{n\pi}{10} x e^{-\frac{n\pi}{10} y} \] is a solution of \( x \) which satisfies 3 of the 4 boundary conditions \(( T(0, y) = T(10, y) = T(x, \infty) = 0) \). To satisfy the 4th boundary condition, we form the linear combination

\[ T(x, y) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{10} x e^{-\frac{n\pi}{10} y} \]
\[ T(x, 10) = 100 = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{10} x \]

(a Fourier sine series of period \(2L = 20\), \(L = 10\))

Thus, for each \(n \geq 1\),

\[ b_n = \frac{2}{L} \int_{0}^{L} f(x) \sin \frac{n\pi}{L} x \, dx \]

\[ = \frac{2}{10} \int_{0}^{10} 100 \sin \frac{n\pi}{10} x \, dx \]

\[ = \frac{200}{10} \left( \frac{-10}{n\pi} \cos \frac{n\pi}{10} x \right) \bigg|_{x=0}^{10} \]

\[ = \frac{200}{n\pi} \left( 1 - \cos \frac{n\pi}{10} \right) \]

\[ = \begin{cases} 
\frac{400}{n\pi} & \text{if } n \text{ is odd} \\
0 & \text{if } n \text{ is even} 
\end{cases} \]

Thus,

\[ T(x, y) = \frac{400}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{n\pi}{10} x e^{-\frac{n\pi}{10} y} \]

\[ = \frac{400}{\pi} \left[ \sin \frac{\pi x}{10} e^{-\frac{\pi y}{10}} + \frac{1}{3} e^{\frac{3\pi}{10} x} e^{-\frac{3\pi y}{10}} + \cdots \right] \]
7. \[ \nabla^2 u = \frac{1}{\Omega} \frac{\partial}{\partial z} \left( \Omega \frac{\partial u}{\partial z} \right) + \frac{\partial^2 u}{\partial z^2} = 0 \]

Let \( u(\Omega, z) = R(\Omega) Z(z) \). Then

\[ \frac{Z(z)}{\Omega} \frac{d}{dz} \left( \Omega \frac{dR}{dz} \right) + R(\Omega) \frac{d^2 Z}{dz^2} = 0 \quad (s \frac{1}{\Omega z}) \]

\[ \Rightarrow \frac{1}{\frac{1}{2}R} \frac{d}{dn} \left( R \frac{dR}{dn} \right) + \frac{1}{2} \frac{d^2 Z}{dz^2} = 0 \]

\[ \frac{1}{\frac{1}{2}R} \frac{d}{dn} \left( R \frac{dR}{dn} \right) = -\frac{1}{2} \frac{d^2 Z}{dz^2} = -\kappa^2 \]

\[ \frac{d^2 Z}{dz^2} - \kappa^2 Z = 0 \rightarrow Z(z) = \left\{ e^{-\kappa z}, e^{\kappa z} \right\} \]

To make \( Z(10) = 0 \), we use

\[ Z(z) = \sinh \kappa (10 - z), \] which is a linear combination of \( e^{-\kappa z} \) and \( e^{\kappa z} \).
To find $R(z)$:

$$\frac{1}{\sqrt{z}} \frac{d}{dz} \left( z \frac{dR}{dz} \right) = -k^2 \Rightarrow$$

$$\frac{1}{\sqrt{z}} \frac{d}{dz} \left( z \frac{dR}{dz} \right) + k^2 R = 0$$

$$\frac{d^2 R}{dz^2} + \frac{1}{z} \frac{dR}{dz} + k^2 R = 0$$

$$\frac{d^2 R}{dz^2} + \frac{1}{z} \frac{dR}{dz} + k^2 z^2 R = 0$$

$$R(\sqrt{z}) = J_0(k \sqrt{z})$$

From $U(1,3) = 0$, we get $R(1) = 0$ and hence $J_0(k) = 0 \Rightarrow k = k_m$, a zero of $J_0$.

To match the boundary condition at the bottom, we write

$$U(2,3) = \sum_{m=1}^{\infty} b_m J_0(k_m \sqrt{z}) \sin \theta k_m (10 - z)$$

$$U(2,0) = U_0 = \sum_{m=1}^{\infty} b_m J_0(k_m \sqrt{z}) \sin \theta k_m (10 km)$$

Multiply both sides by $2 J_0(k_m \sqrt{z})$ and integrate from 0 to 1.
\[ 100 \int_0^\infty x J_0(k_x x) \, dx = \sum_{m=1}^{\infty} b_m \sinh(10k_m) \int_0^\infty x J_0(k_x x) J_0(k_m x) \, dx \]

\[ = b_\mu \sinh(10k_\mu) \frac{J_1^2(k_\mu)}{2} \quad (**) \]

But \[ \int x J_0(x) \, dx = -\frac{1}{2} J_1(x) \quad \text{(with } x = k_x x) \]

we get \[ k_\mu^2 \int x J_0(k_x x) \, dx = k_\mu x J_1(k_x x) \]

Therefore \[ \int_0^1 x J_0(k_x x) \, dx = \frac{1}{k_\mu} \frac{x J_1(k_x x)}{x} \Bigg|_{x=0}^{x=1} \]

\[ = \frac{J_1(k_\mu)}{k_\mu} \]

\[ (***) \Rightarrow 100 \frac{J_1(k_\mu)}{k_\mu} = b_\mu \sinh(10k_\mu) \frac{J_1^2(k_\mu)}{2} \]

\[ \Rightarrow b_\mu = \frac{200}{k_\mu \sinh(10k_\mu) J_1(k_\mu)} \quad \text{or} \]

\[ b_m = \frac{200}{k_m \sinh(10k_m) J_1(k_m)} \quad \text{for } m \geq 1. \quad \text{Thus,} \]

\[ \nu(\alpha, \beta) = \sum_{m=1}^{\infty} \frac{200}{k_m \sinh(10k_m) J_1(k_m)} J_0(k_m x) \sinh(k_m x) \]

\[ k_m \text{ is the } m^{th} \text{ zero of } J_0. \]