"The Library of Magical Squares" - a summary of the main results for the Shannon entropy of magic and Latin squares: isentropic clans and indexing, in celebration of George Styan's 75th. Movie at LINSTAT2012 and IWMS-21 on 19 July, 2012 at Bewedlo, Poland.

Ian Cameron, Adam Rogers and Peter Loly,
Department of Physics and Astronomy, University of Manitoba,
Winnipeg, Manitoba, Canada R3T 2N2

19 July 2012

Abstract

The 2010 study of the Shannon entropy of order nine Sudoku and Latin square matrices by Newton and DeSalvo [PRSA 2010] **is extended** to a comprehensive survey of natural magic squares for orders three to nine. Complete sets of magic squares at orders four and five are included, as is the order eight Franklin subset. Comparison across different orders is illuminated by using **an effective rank measure.**

While early examples suggested that lower rank specimens had lower entropy, sufficient data is presented to show that some full rank cases with low entropy possess a set of SVDs separating into a dominant group with the remainder much weaker. An effective rank measure helps understand these issues.

We also introduce a new index for integer squares based on the sum of the fourth powers of the singular values which appears to give a useful method of indexing both Latin and magic squares. This is useful for the "library".

Keywords: Shannon entropy; magic square; Latin square; singular value decomposition.

[short title for page header: Magic square entropy]

For software advice for implementing the ideas presented contact:

icamern@ad.umanitoba.ca: FreeMat, Macsyma, Python, Octave;
rogers@physics.umanitoba.ca: MATLAB, Maple;
loly@cc.umanitoba.ca: Mathematica, Maple, Wolfram Alpha via web browser or iPad.

1 Introduction

We study the Shannon entropy [22] of sets of natural magic square matrices (order $n > 2$), which have the same line sum of elements in all rows, columns and both main diagonals, with full (or exact) cover [2] of the elements $1..n^2$ so that all elements are distinct. Hereafter we refer to the natural variety simply as magic squares. Semi-magic squares lack one or both of the diagonal sums. Latin squares do not use this constraint. Magic, semi-magic and Latin squares are all doubly affine [24] integer matrices, and as such often exhibit more elegant results than real number matrices, e.g. their determinant vanishes in singular cases.

The Shannon entropy of Sudoku matrices was studied by Newton and DeSalvo [13]. Newton and DeSalvo [13] do not mention magic squares, yet much of what they examine for 9th order Latin and Sudoku squares is immediately applicable to smaller Latin squares, briefly included here **for the insight they give into both larger Latin squares**, as well as a wide range of magic squares.

Following Newton and DeSalvo [13] we also calculate a percentage of compression factor for the reduction of the entropy from a reference maximum entropy which goes as the logarithm of the order of the squares, $\ln n$, in order to provide a comparison across different orders. Newton and DeSalvo [13] found average compressions in the range of 21 to 25%.

The essential input to the calculation of the Shannon entropy is the singular value decomposition (SVD) of the matrices. The number of non-zero SVs (σ_i) gives the rank of a matrix. These SVs were recently studied for selected magic squares from order 3 to order 8 by Loly, Cameron, Trump and Schindel [9] in order to help understand the complicated behaviour of their eigenvalues, but since the main focus concerned the eigenvalues (λ_i), only selected SV values were reported. The number of magic squares studied here is larger and now includes the smallest compound magic squares (order 9) which are highly singular **and low rank**. SVs give an advantage over the eigenvalues since the SVs are invariant under rotation and/or reflection of the matrices [8][9], as well as tiling to semi-magic squares. Although the SVs are usually listed as non-increasing, we have found many degenerate (equal) values in studies of small Latin squares, as well as a few for magic squares.

Semimagic squares have the same linesum for all Rows and Columns, which we call RC-symmetry. Magic squares also have the same linesum for the principal diagonals $D1$ and $D2$ which we call RCD-symmetry. However only some natural Latin squares and no natural magic squares are symmetric matrices.

For matrix A the eigenvalues of the covariance matrices AA^T and A^TA, also known as Gramian matrices**,** give the squares of the singular values [9]. Both AA^T and A^TA are symmetric matrices (they are their own transposes), but in general they differ. However they share the same characteristic equation, and thereby the same eigenvalues.

While there is only one magic square of order three (the ancient Chinese Loshu), with a remarkably small compression of 14%, there are complete sets for orders four and five, with 880 and $275,305,224$ distinct members respectively.

We treat these **two** sets as ensembles for the purpose of calculating the mean Shannon entropy, as well as finding their distributions, and particularly the minimum and maximum values. **Much of our focus is on individual squares deemed interesting for their extreme values of Shannon entropy.** For higher orders the populations are so large that they are only known statistically (Trump [27],[9]), except in special cases such as the 8th order Franklin squares counted in 2006 by Schindel, Rempel and Loly [19].

It follows that Shannon entropy is a very useful metric for comparing different magic squares. The same applies also to Latin squares, including solutions of Sudoku puzzles.

1.1 Overview of this work:

We begin with an explicit calculation of the Shannon entropy for the smallest magic square in section 2. **Since sufficient data is later presented to show that some full rank cases with low entropy possess a set of SVs separating into a dominant group with the remainder much weaker, an effective rank measure is introduced here to help understand these issues. We also present a wider range of Sudokus for comparison with Newton and De Salvo [13].** Then a discussion of the characteristic polynomials for matrix eigenvalues and singular values is given in section 3.1. We introduce some integer measures in section 3.2 that are useful in keeping track of individual examples **and offer a method of indexing integer squares.** After examining general aspects of magic squares, the complete set of magic squares of order four is examined in new detail in section 5 The complete set of magic squares in order five has been examined. **Special attention is paid to the cases of 3 non-zero eigenvalues (rank 3) for which calculations simplify.**

As far as possible we use the same notation as Newton and DeSalvo [13] in what follows.

2 Calculation of Shannon entropy for the ancient Loshu magic square

We set the scene for several issues by first examining the sole natural magic square of order three:

$$LoShu = \begin{bmatrix} 4 & 9 & 2 \\ 3 & 5 & 7 \\ 8 & 1 & 6 \end{bmatrix}, \tag{1}$$

characteristic polynomial: $x^3 - 15x^2 + 24x - 360$, where 15 is the linesum eigenvalue (and trace, λ_1), 360 is the determinant, and 24 is the sum of the determinants of the three 2-by-2 principal minors:

$$\det\begin{bmatrix}4 & 9\\3 & 5\end{bmatrix} = -7, \det\begin{bmatrix}5 & 7\\1 & 6\end{bmatrix} = 23, \det\begin{bmatrix}6 & 8\\2 & 4\end{bmatrix} = 8,\ \text{sum: } 24 \tag{2}$$

The Loshu exhibits full cover of the matrix elements 1..9 in a manner such that all antipodal pair sums about the centre yield 10, so that it belongs to the type of magic square called regular (or associative). The Loshu has eigenvalues $\lambda_1 = 15$, the Perron root for a positive square matrix, and a signed pair: $\pm 2i\sqrt{6}$ [9]. Here the sum of the eigenvalues is λ_1, an example of the fact that $\sum_{i=1}^{n} \lambda_i = \lambda_1$ for all magic squares [9].

The matrix products AA^T and A^TA are symmetric matrices since they are their own transposes, but in general they differ. However they share the same characteristic equation, and thereby the same eigenvalues. Taking here $A = Loshu$ from (1):

$$AA^T = \begin{bmatrix}101 & 71 & 53\\71 & 83 & 71\\53 & 71 & 101\end{bmatrix}, A^TA = \begin{bmatrix}89 & 59 & 77\\59 & 107 & 59\\77 & 59 & 89\end{bmatrix}, \tag{3}$$

which both have the same eigenvalues $\left(\sigma_i^2\right)$ since in general these products always have identical characteristic polynomials:

$$x^3 - 285x^2 + 14\,076x - 129\,600;\ \sigma_i^2 = 225, 48, 12 \tag{4}$$

Observe that the sum of these eigenvalues is an integer: $\sum_{i=1}^{n} \sigma_i^2 = 285$. In fact all natural squares, not just magic squares, have an integer sum of the squares of the SVs since this is an invariant for each order. While generally AA^T and A^TA are each symmetric, here they are bisymmetric.

The eigenvalues of AA^T are the squares of the SVs, with the σ_i listed here in non-decreasing order ($\sigma_i \geq \sigma_j$, $i < j$):

$$\sigma_1 = 15, \sigma_2 = 4\sqrt{3}, \sigma_3 = 2\sqrt{3}, \tag{5}$$

where σ_1 is the same as the trace of the magic square, which equals λ_1 since the other eigenvalues add to zero [9]. The sum of the (positive) singular values is always greater than the linesum eigenvalue, including the unusual non-diagonable cases [9].

We now follow Newton and DeSalvo [13] and normalize the σ_i by their sum:

$$\hat{\sigma}_i = \frac{\sigma_i}{\sum \sigma_i}, 0 \leq \hat{\sigma}_i \leq 1, \tag{6}$$

so that for the Loshu**:** $\hat{\sigma}_1 = 0.590730148$, $\hat{\sigma}_2 = 0.272846568$, $\hat{\sigma}_3 = 0.136423284$. The decimal values will usually be rounded to six or seven digits, unless otherwise stated.

Then we obtain the Shannon entropy (H) as per Newton and DeSalvo [13]:

$$H = -\sum_i \hat{\sigma}_i \ln\left(\hat{\sigma}_i\right), \tag{7}$$

finding $H = 0.9370977555$. The descending (non-increasing) SVs contribute differently to the entropy: 0.31096, 0.35439, 0.27175, respectively, with the largest contribution from the second SVD in this case. For order 5 we found an order five magic square where σ_1 has the larger contribution. $rank = \exp(0.9370977555) = 2.5526$.

Finally we find the percentage compression, C, by generalizing Newton and DeSalvo's [13] for the $n = 9$ Sudoku case to reference $\ln(n)$ instead of their $\ln(9)$**:**

$$C = \left(1 - \frac{H}{\ln(n)}\right) \times 100\%, \tag{8}$$

finding $C = 14.70168640\%$. Hereafter we usually round off the compression to two decimal places, e.g. 14.70%. Compression varies oppositely to the entropy for a given order, and $\frac{H}{\ln(n)} \times 100\%$ is the complement of C.

2.0.1 Effective rank

An effective rank measure [18] which is found to be useful later is simply:

$$erank = \exp(H), \tag{9}$$

where for the rank three Loshu (1)**,** $erank = 2.55256$, **the reduction from full rank reflecting the decreasing magnitudes of the SVs.**

Note that it suffices to list either just one of H**,** $erank$**, or** C, **since** the others can be obtained **when** n **is known. Here** $erank$ is preferred.

3 Magic squares

The number of distinct magic squares grows rapidly from the unique third order Loshu, through 880 at fourth order and $275,305,224$ at 5th order, with only statistical estimates, available for $n > 5$ [27]. Now the linesum eigenvalue is:

$$\lambda_1 = S_n = n(1 + n^2)/2 = \sigma_1 \tag{10}$$

For magic squares a_1 in (13) is just λ_1 since the rest sum to zero [9].

There are five particularly important types of magic squares: associative (also called regular), pandiagonal, ultramagic, bent diagonal (or Franklin) [19], and compound.

Associative (or regular) have elements antipodal about the centre with the same pair sum, with the Loshu (1) as an example:

$$a_{ij} + a_{n-i+1,n-j+1} = (1 + n^2),\ i, j = 1, ..., n \tag{11}$$

In pandiagonal (also called Nasik) squares the broken diagonals (n consecutive elements parallel to the main diagonals under tiling, or periodic boundary condtions have the same sum as the main diagonals. Note that the addition tables in the previous section are associative and pandiagonal, but not magic.

Ultramagic squares have both the associative and pandiagonal properties. For singly even orders, e.g. $n = 6, 10, 14, ...$there are no associatives or pandiagonals, and so no ultramagics.

Franklin's famous squares [19] for $n = 8, 16$ have 'bent' diagonals with elements adding to the magic sum, but not necessarily the main diagonals. They also a fixed sum for the half rows and columns, and with all 2-by-2 quartets having a fixed sum. They are interesting here because they exhibit the minimum rank of three.

Compound magic squares (CMSs)[17] of order $n = pq$ have order p or q tiled magic subsquares. These begin at $n = 9$ and are interesting because they have lower ranks than all but Franklin type magic squares when they share the same doubly even order ($n = 4, 8, 12, 16, 20, 24, ...$).

3.0.2 Other measures for numerical squares

We now examine another function of the SVs which is useful for keeping track of numerical squares which will later be a useful adjunct to H for all integer squares. For now we begin with general square matrices.

3.1 Characteristic polynomials and symmetric functions

Consider a square matrix, A, of order n, with eigenvalues λ_i. Then factor the general nth order characteristic polynomial to show its n roots:

$$\alpha = x^n - a_1 x^{n-1} + a_2 x^{n-2} - ... \pm a_n = (x - \lambda_1)(x - \lambda_2) ... (x - \lambda_n) = 0. \quad (12)$$

Girard [6] [11] [5] (in 1629, more than two centuries before the matrix theories of Cayley and Sylvester) showed that the first coefficient a_1 in (12) gives the sum of the roots (the trace of the matrix eigenvalues here):

$$G1 = \sum_{i=1}^{n} \lambda_i = a_1 \ . \quad (13)$$

The sum of the squares of the roots involves both a_1 and a_2:

$$G2 = \sum_{i=1}^{n} \lambda_i^2 = a_1^2 - 2a_2. \quad (14)$$

where a_2 is the sum of the determinants of all 2-by-2 principal minors of A, i.e. $a_2 = \prod_{i<j} \lambda_i \lambda_j$ [9].

These identities can be obtained by squaring (12). Girard [6] also gave a formula for the cubes:

$$G3 = \sum_{i=1}^{n} \lambda_i^3 = a_1^3 - 3a_1 a_2 + 3a_3, \quad (15)$$

and for 4th powers which are not needed in this work. Later these identities were rediscovered by Newton, and often known as Newton's method.

3.2 Covariance matrices and application to singular values

Since the SVs are the square roots of the eigenvalues of AA^T (or A^TA) [8] they must satisfy a variant of (12):

$$\beta = x^n - b_1 x^{n-1} + b_2 x^{n-2} - ... \pm b_n = \left(x - \sigma_1^2\right)\left(x - \sigma_2^2\right)...\left(x - \sigma_n^2\right) = 0. \quad (16)$$

For the SVs, applying (13) and (14) to (16) means that the sum of the squares of the SVs is given by:

$$P1 = \sum_{i=1}^{n} \sigma_i^2 = b_1, \quad (17)$$

and then the counterpart of (14) gives the sum of the fourth powers of the SVs:

$$P2 = \sum_{i=1}^{n} \sigma_i^4 = b_1^2 - 2b_2, \quad (18)$$

where b_2 is the sum of the determinants of all 2-by-2 principal minors of AA^T, i.e. $b_2 = \prod_{i<j} \sigma_i \sigma_j$. The counterpart of (15) for the sum of the 6th powers finds an application later in section **??**:

$$P3 = \sum_{i=1}^{n} \sigma_i^6 = b_1^3 - 3b_1 b_2 + 3b_3, \quad (19)$$

It is worth noting a connection with Schatten $p-$norms and Ky Fan's $p-k$ norms, both of which include even and odd powers p of the SVs, rather than the even powers which flow from our use of Girard's results [6]. However the results for these norms are usually not integers since they involve the $p-$th roots of the sums of powers p of the SVs. Ky Fan's k-norm of A uses the k largest singular values of A, so that his 1-norm is the largest singular value of A, while the last of his norms, the sum of all singular values, is the trace norm. The Schatten 2-norm is the square root of the squares of all the singular values of A.

3.3 A further step (2012)

Now factor out $\left(x - S_n^2\right)$ or $\left(x - \sigma_1^2\right)$ from (16): $\beta = \left(x - \sigma_1^2\right)\gamma$

$$\gamma = x^{n-1} - d_1 x^{n-2} + d_2 x^{n-3} - ... \mp d_{n-1}] = \left(x - \sigma_2^2\right)...\left(x - \sigma_n^2\right) = 0. \quad (20)$$

where

$$d_1 = b_1 - \sigma_1^2,\ d_2 = b_2 - \sigma_1^2 d_1,\ \text{etc.} \tag{21}$$

and now

$$d_1 = \sum_{i=2}^{n} \sigma_i^2,\ d_2 = \sum_{i=2}^{n-1} \sum_{j>i}^{n} \sigma_i^2 \sigma_j^2, .., \text{etc.} \tag{22}$$

4 Integer matrices: magic and Latin squares

Now that the matrix elements are integers, the a_i coefficients are integers in (12), (13), (14), (15), as are the b_i coefficients in (16), (17), (18), and () We have already seen from section 2 that $G1$ and $P1$ are integers for the Loshu. Appendix A shows that $P1$ has an n-dependent value for all natural squares of any order. Now it is clear from (18) that $P2 = \sum_{i=1}^{n} \sigma_i^4$ is also an integer**,** even though we can show that individual σ_i^2 are not always integer.

4.1 Long and Short indices

Since b_1 and b_2 are now integers, $P2$ (18) gives a Long integer index, L, for all integer squares:

$$L\,[Loly] = \sum_{i=1}^{n} \sigma_i^4 = b_1^2 - 2b_2 = \sigma_1^4 + d_1^2 - 2d_2, \tag{23}$$

and for doubly affine natural squares (e.g. Latin, magic or semi-magic), where $\sigma_1 = \lambda_1$ is also an integer, there is a Reduced integer index, R**:**

$$R\,[Rogers] = L - \sigma_1^4 = \sum_{i=2}^{n} \sigma_i^4 = b_1^2 - 2b_2 - \lambda_1^4 = d_1^2 - 2d_2. \tag{24}$$

So while index L appears to be useful for indexing all integer squares, index R offers a shorter reduced index for doubly affine integer squares. For the Loshu (1) $L = 53073$ and $R = 2448$. N**ote that** R, as the sum $\sum_{i=2}^{n} \sigma_i^4$, **is independent of the choice between using elements** $1..n^2$ **or** $0..\left(n^2 - 1\right)$**. One might consider using** b_2 **as an index since** b_1^2 **and** λ_1^4 **depend only on** n **however numerically** $b_2 > R$**, and the smaller index is preferable. In any case the** b_i **depend on the choice of elements.**

A third integer key, Q, which can be useful in resolving degeneracies in R (and L) is obtained from the sixth powers in (19):

$$Q\,[Cameron] = \sum_{i=2}^{n} \sigma_i^6 = b_1^3 - 3b_1 b_2 + 3b_3 - \lambda_1^6 = d_1^3 - 3d_1 d_2 + 3d_3 \tag{25}$$

Note that it is recommended that the integer $d_1, d_2, d_3, ...$ be used to obtain R, L, Q to avoid rounding errors in computed values of the SVs may enter the direct calculation of sums of powers of the SVs.

The integer measures, L and R, vary amongst (integer) Latin and magic squares of a given order, and those for different orders are separated by large gaps. Duplicate (degenerate) values occur when transformations, e.g. certain row-column permutations, produce a distinct magic square with the same H - these isentropic sets will be called clan members (of the ith clan). Later we find some cases where the same key occurs for different clans, i.e. distinct values of H, e.g. we find one order four example of the same L, R for different H.

Index R can now be computed for the last two rows of Table 1 giving $63,408$ and $67,068$ respectively. Also for Table 2 the respective values are: $119,556$, $100,692$, $94,644$, $82,753$, $76,935$ and $40,824$.

4.2 Overview of magic squares as order increases

Figure 1 shows the results of using the MATLAB MAGIC(n) function **(we use *magicn* to identify these in later tables)**, together with the upper bound of $\ln n$ for $n = 3..26$. MATLAB [10] uses one algorithm to produce non-singular (full rank) associative magic squares for odd order (the squares in Figure 1), and a second algorithm for singular even order magic squares (the squares in Figure 1) which are not associative but have rank $(n+4)/2$. MATLAB's third algorithm gives a family of rank 3 (singular) doubly even magic squares (the inverted triangles in Figure 1).

Figure 1 suggests that higher rank magic squares have the **higher entropies** (and rather disordered elements), while rank three magic squares have the **lower entropies** (and rather ordered elements), with intermediate rank magic squares **with intermediate values**.

For each type in Figure 1 the approximate compressions at the right hand side are: doubly-even (inverted triangles) $n = 100$ with $C = 85.27\%$; even (circles) $n = 98$ with $C = 54.71\%$; and odd (squares) $n = 99$ with $C = 23.09\%$.

Figure 3 plots Shannon entropy versus the index R for the order four set of 63 clans. Note the degeneracy at $R = 86728$.

5 Complete set of order 4 magic squares

Figure 2 shows the 63 clans for order 4. The singular clans lie on the lower curve, the non-singular above, and note that there is a degeneracy in $R = 86728$ for two clans.

Many such degeneracies occur for order 6 Latin squares, e.g., for standard order Latin squares there are four clans with $R = 4689$.

6 "The Library of Magic Squares"

R is used to find the correct shelf, while Q the proper position on that shelf.

7 Acknowledgements

Jason Fiege, Marcus Steeds, Wayne Chan, Daniel Schindel, George Styan, Harvey Heinz, Walter Trump, Francis Gaspalou, Russell Holmes, Jayanne English, the Winnipeg Foundation in 2003, Craig Knecht and many water retention collaborators), Robert Thomas , Paul Pasles, Dwayne Campbell, ...

References

[1] Andrews, W.S. 1908 *Magic Squares and Cubes*, 1st edition, The Open Court Publishing Company, New York, 1914 2nd edition with corrections and added chapters. [http://www.magic-squares.net/square-update.htm#W.%20S.%20Andrews%20error]

[2] Brualdi, R., Introductory Combinatorics, 5th edition, Pearson, 2010

[3] Cameron, I. & Loly, P. 2009 Eigenvalues of an Algebraic Family of Compound Magic Squares of Order $n = 3^l, l = 2, 3, ..,$ and Construction and Enumeration of their Fundamental Numerical Forms, Canadian Mathematical Society, Windsor, Ontario.

[4] Eggermont, C. 2007 Multimagic Squares, Thesis, Department of Mathematics, Radbout University of Nijmegen.

[5] Funkhouser, H.G. 1930 A Short Account of the History of Symmetric Functions of Roots of Equations, Am. Math. Monthly 37(7) 357-365.

[6] Girard. A. 1629 *Invention nouvelle en l'algèbre*, gallica.bnf.fr/ark:/12148/bpt6k5822023w.r=langFR

[7] Loly, P., 2007 Franklin Squares - A Chapter in the Scientific Studies of Magical Squares, (version of a poster talk at NKS2006 (New Kind of Science), Washington, D.C., June 2006), Complex Systems, vol. 17, 143-161.

[8] Loly, P.D. 2008 Two small theorems for square matrices rotated a quarter turn, Western Canada Linear Algebra Meeting (WCLAM2008), Winnipeg.

[9] Loly, P., Cameron, I., Trump, W. & Schindel, D. 2009 Magic square spectra, Linear Algebra and Its Applications, 430 2659-2680.

[10] Moler, C. 1993 MATLAB's Magical Mystery Tour, The MathWorks Newsletter, 7(1) 8-9.

[11] Miller, G.A. 1916, 1927 Historical Introduction to Mathematical Literature, The MacMillan Company.

[12] Miller, G.A. 1909 Review of [1], Bull. Amer. Math. Soc. 16: 85-87.

[13] Newton, P.K. & DeSalvo, S.A. 2010 The Shannon entropy of Sudoku matrices, Proc. R. Soc. A 466:1957-1975.

[14] Ollerenshaw, Dame K. 1986 On 'most perfect' or 'complete' 8x8 pandiagonal magic squares. *Proc. R. Soc. Lond.*, **A407**, 259-281.

[15] Ollerenshaw, K. and Brée, D. S. 1998 Most-perfect pandiagonal magic squares: their construction and enumeration, The Institute of Mathematics and its Applications, Southend-on-Sea, UK.

[16] Pasles, P.C. 2008 Benjamin Franklin's Numbers: An Unsung Mathematical Odyssey, Princeton.

[17] Rogers, A. 2004; Rogers, A., Loly, P. & Styan, G.P.H. 2008 Sums of Kronecker Products for Compound Magic Squares: Eigenproperties, Western Canada Linear Algebra Meeting (WCLAM2008), Winnipeg.

[18] Roy, O. and Vetterli, M. 2007 The Effective Rank: A Measure of Effective Dimensionality, EUSIPCO (EURASIP), Poznań.

[19] Schindel, D. G., Rempel, M. & Loly, P. D. 2006 Enumerating the bent diagonal squares of Dr Benjamin Franklin FRS, *Proceedings of the Royal Society A: Physical,* Mathematical and Engineering, 462, 2271-2279

[20] Setsuda, K., 7×7 Self-Complementary Pan-magic Squares: mathforum.org/te/exchange/hosted/suzuki/MagicSq.7x7.selfsim.pan.html.

[21] Schroeppel, R. 1971 The order 5 magic squares. Written by Michael Beeler with assistance from Schroeppel - see M. Gardner's 1976 column on Mathematical Games, *Scientific American* **234**, 118-123.

[22] Shannon, C. [See NDS]

[23] N. J. A. Sloane, 2012, The On-Line Encyclopedia of Integer Sequences http://www.research.att.com/~njas/sequences/

[24] Thompson, A.C., Odd magic powers, Am. Math. Monthly, 101 (1994) 339-342

[25] Thompson, W.H. 1869 On magic squares, The Quarterly Journal of Pure and Applied Mathematics, X 186-202 [XI, 57, 123, 213?]

[26] Trump, W. 2003 *Notes on Magic Squares and Cubes* http://www.trump.de/magic-squares/

[27] Trump, W. 2003 Estimate of the number of magic squares of order 6. http://www.trump.de/magic-squares/normal-6/index.html

[28] Weisstein, E. W. *Magic Square(s)*. From MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/MagicSquare.html

1 Electronic files

1. TBA

2 List of Tables and Figures

Table 2 - Sudokus from rank 5 to rank 9. ($\sigma_1 = 45$)

r[a]	square[b]	H	C (%)	R[d]	$erank$[e]
9	$idc15m$	1.2288	44.08	$142,236$	3. 41712
5	$idc5p$[g]	1.2952	41.05	$119,556$	3.65182
6	$idc6p$	1.4296	34.94	$100,692$	4.17710
9	$knecht$	1.5610	28.96	$101,028$	4. 76339
7	$idc7p$	1.5389	29.96	$94,644$	4.65925
8	$idc8p$	1.6406	25.33	$82,752$	5.15814
9	$bailey12$	1.6700	23.99	$76,208$	5.31231
8	$nds21$	1.7119	22.09	$66,864$	5.53922
9	$idc9p$	1.7127	22.05	$76,936$	5. 54363
9	$bailey11$	1.7610	19.85	$62,636$	5.81851
9	$ndsfig1$	1.7789	19.04	$63,408$	5.92335
9	$nds22$	1.8163	17.34	$67,068$	6.14895
9	$idc69$	1.8878	14.05	$40,824$	6.60506
9	$idc14m$	1.9083	13.15	$36,936$	6.74167

Table 6: Order four magic squares [$\ln(4) = 1.386\,3$]

r	square		F[a]	clan[b]	H	C (%)	R	$erank$
3	G1-3, α	high[d] C	116	9		37.23	102800	2.38739
3	G1-3, β	middle	107	35		31.56	78608	2.58246
3	G1-3, γ	low[e] C	102	41		30.88	74000	2.60697
3	G1-3	mean[f]				33.22	85136	
3	G4-6-P	**MAX**[g] C, R	25	1		42.02	111376	2.2339
3	G4-6-P	low C	73	40		30.90	74128	2.60632
3	G4-6-P	mean				35.47	92752	
3	G6-S	high C	51	4	.828470568	40.24	109000	2.28981
3	G6-S	low C	46	62	.985000976	28.95	58000	2.67781
3	G6-S	mean			.939108	32.26	78319.7	
3	S-pair[h]	(G6-S)	$1,714$	26	.929245522	32.97	86728	2.5326
4	S-pair[h]	(G8)	278	25	1.08097925	22.02	86728	2.94756
4	G7-10	high C	10	3	.904033853	34.79	109264	2.46954
4	G7-10	**MIN**[i] C, R	775	63	1.12861366	18.59	57232	3.09137
4	G7-10	mean			1.03155	25.59	80821.4	
4	G11,12	low C	3	19	1.01521653	26.77	93584	2.75996
4	G11,12	min C	88	42	1.09252315	21.19	73424	2.98179
4	G11,12	mean			1.05387	23.90	83504	
#	full 880 set	mean			.947977	31.62	84889.5	

Table 8 - $n = 5$ [entropy increases down column 5, $\ln(5) = 1.6094$.]

r	square	H	$C\,(\%)$	R	$erank$
4	min[H]	0.9142	43.2		
4	$lcts43m2$[c]	1.05067	34.72	$1,218,640$	2.85955
5	$suz6, 9$[d]	1.12526	30.08	$954,480$	3.08101
5	$suz2, 13$[d]	1.12706	29.97	$904,500$	3.08658
4	$lcts44m4$	1.20122	25.36	$706,000$	3.32418
5	$suz5, 10$[d]	1.20354	25.22	$822,000$	3.33188
5	$lcts45, suz1$	1.23161	23.48	$772,980$	3.42674
$4, 5$	**full set**	1.2827	20.3		
5	$suz7, 12$[d]	1.37667	14.46	$522,480$	3.96171
5	$suz3, 16$[d]	1.3783	14.36	$544,500$	3.96813
5	$suz8, 11$[d]	1.38085	14.20	$582,000$	3.97827
5	$suz4, 15$[d]	1.38229	14.11	$604,980$	3.98403
5	$magic5$	1.38932	13.68	$532,000$	4.01212
5	max[H]	1.442	10.4		

Table 11 $n = 8$, [$\ln(8) = 2.0794$]

r	square	H	$C\,(\%)$	R[a]	$erank$
3	$bf8minhSM$[b]	.70720	65.99	$476,125,440$	2.0283
3	$magic8$ & many others	.80335	61.37	$462,534,912$	2.23301
3	$dko86$ most-perfect, complete	.87072	58.13	$431,200,512$	2.38862
4	$wht8a$[f]	.91569	55.96	$431,560,960$	2.4985
4	$wht8b$[f]	.94361	54.62	$429,496.576$	2.56923
3	$dc8$[e]	.96009	53.83	$323,096,832$	2.61194
3	$bf81SM$[b]	.96816	53.44	$306,950,400$	2.63309
3	$pbf151$[b]	.96942	53.38	$304,296,192$	2.63641
3	$bf8maxh$	.97144	53.28	$299,964,672$	2.64173
5	$fz8$	1.1984	42.37	$294,752,512$	3.31495
7	$frolow$[i]double	1.3351	35.80	$286,331,136$	3.80037
7	$gasp13$[h]	1.2867	38.12	$224,354,880$	3.62073
8	$i88e8$	1.5480	25.56	$283,664,192$	4.70214
7	$seki8$ nested[j]	1.3367	35.72	$235,171,920$	3.80631
7	$bi8a$ Pfefferman	1.5251	26.66	$152,375,424$	4.59565
7	$wtREG001BI$	1.5669	24.65	$163,243,244$	4.79176
5	$euler8SM$[k]	1.4157	31.92	$136,171,776$	4.11942
7	ps, $H*$	1.5381	26.03	$143,056,292$	4.66573
8	$gasp1$[h]	1.6441	20.94	$135,335,220$	5.17623
8	$eulerK8SM$	1.6887	18.79	$146,064,640$	5.41233
7	$wtREG841BI$	1.6829	19.07	$102,971,488$	5.3812

Table 12, n=9 [$\ln(9) = 2.1972$]

r	square	H	$C\,(\%)$	R	$erank$
9	$arno173$	1.20501	45.16	$1,307,982,296$	3.33679
5	ta, td	1.12999	48.57	$1,301,165,856$	3.09562
5	$planck9$	1.17113	46.70	$956,739,600$	3.22563
9	$stifel$	1.37807	37.28	$963,270,240$	3.96724
5	tc, tf	1.32208	39.83	$842,630,688$	3.75121
5	tb, te	1.31781	40.02	$797,281,056$	3.73521
5	$dko9a$	1.33486	39.25	$788,778,000$	3.79948
5	$hauck$	1.33579	39.21	$786,914,676$	3.80298
9	$pf(eff)9BI$	1.55931	29.03	$783,193,032$	4.75553
9	$fr9REG$	1.70913	22.21	$722,958,040$	5.52416
9	va^{e}	1.75091	20.31	$466,097,304$	5.75984
7	$dko9b$	1.69577	22.82	$413,322,912$	5.45082
9	$luna$	1.85005	15.80	$472,695,264$	6.36014
9	$magic9^{\mathrm{z}}$	1.85479	15.69	$455,689,152$	6.3756

Table 13: n>9

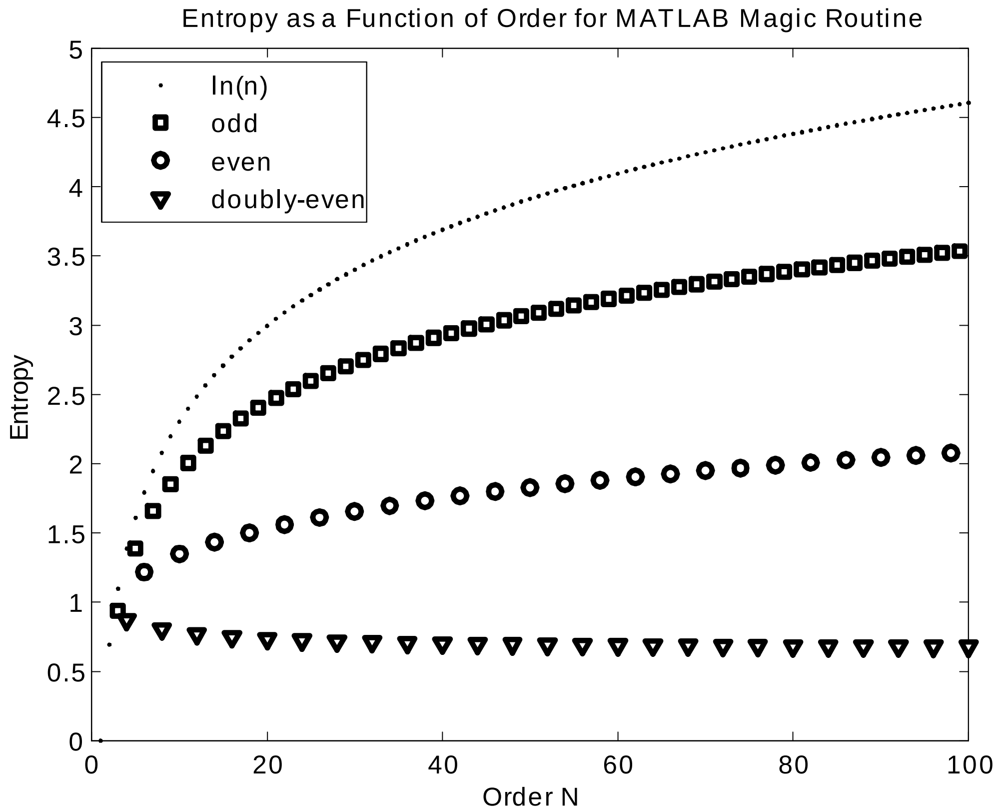
Table 13:

n	r	square	H	$C\,(\%)$	R	$erank$
16	3	$F16Franklin$	.7478348172	73.03	$1,939,654,021,120$	2.11242
12	3	$DMorris"Franklin"$	.7688774508	69.06	$61,063,331,472$	2.15734
12	7	$Trumptrimagic$	1.6678	32.88	$14,239,432,800$	5.3005
16	15	$Pasles$	1.782194364	35.72	$526,731,025,624$	5.94288

3 List of Figures

Figure 1: Entropy as a function of order n for MATLAB routine magic(n)
Figure 2: H for Groups

[used file Acer, f880.eps]
Figure 3: H v. S for n=4



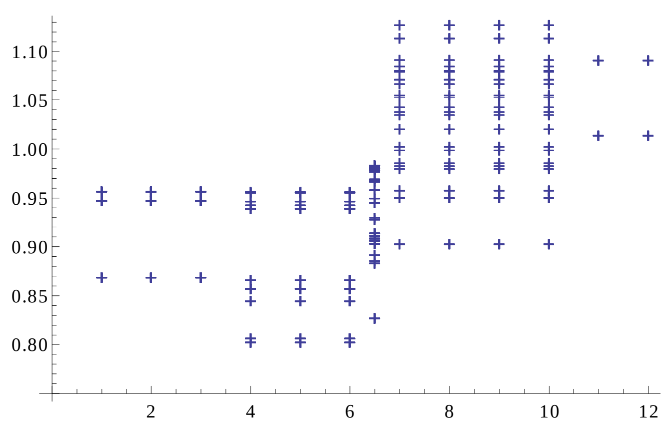


Figure 1: H-values for Dudeney's groups - channel 6.5 used for subgroup 6-S.

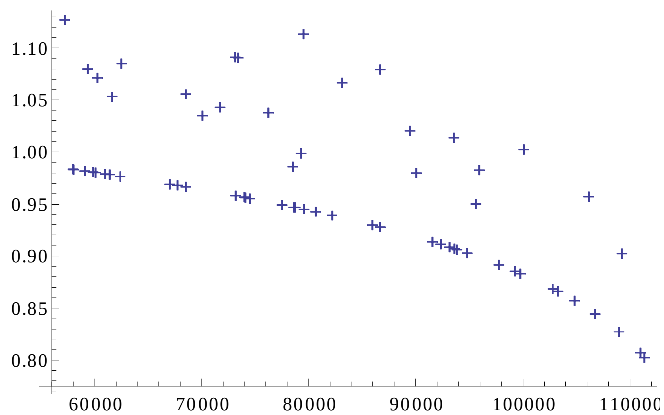


Figure 2: H versus R. The data is double valued in H at $R = 86728$.