Topology and Physics

• Topology is a branch of mathematics concerned with geometric configurations that are unchanged by elastic deformations or twists.

• A topologist cannot tell the difference between a coffee cup and a donut!

• One can be continuously deformed into the other by stretching and indenting the surface without tearing it.

• Topological equivalence can only be destroyed by a drastic change such as tearing or gluing parts together.

Coffee Cup = Donut

Topology and Knot Theory

• the Möbius Strip has only one side and one edge!

Möbius Strip

Has applications to quantum computing and DNA replication
One sided? One edge?

• (1) Start midway between the "edges" of a Möbius Strip and draw a line down its center; continue the line until you return to your starting point without lifting the pen. Did you ever cross an edge?
• (2) Next, hold the edge of a Möbius Strip against the tip of a felt-tipped highlighter pen. Color the edge of the Möbius Strip by holding the highlighter still and just rotating the Möbius Strip around. Were you able to color the entire edge?
• (3) Now, with scissors cut the Möbius Strip along the center line that you drew. Then draw a center line around the resulting band, and cut along it. Did you predict what would happen?

Quantum Numbers

• The different values of "k" can be used to classify the topology of the strips
• These are conserved quantities like "charges"
  • half integer strips are one sided and integer strips are two sided
• In quantum mechanics, angular momentum is conserved and can take integer or half-integer values
  • Half-integer values describe fermions (electrons, protons, neutrons, quarks)
  • Integral values describe bosons (4He, photons)
• In quantum mechanics we can classify particles in a similar way to that we used to classify strips

Symmetry

• What distinguishes different phases of matter?
• Phases differ in their symmetry
• Which is more symmetric?

A cube breaks rotational symmetry => some directions are more equal than others

Which is more symmetric?

• Ice can be rotated by 120° or 240°
• Breaks translational symmetry => can only be shifted by special distances
• No symmetry at all!
• Complete rotational and translational symmetry

• How do we tell if two materials differ by a symmetry?
• Can we change one into the other smoothly by changing temperature?
• Ice turns to water at a phase transition
What is the order parameter for a magnet?

- At each point \((x,y,z)\) we have a magnetization vector \(M(x)\)
- In becoming a magnet, the material has broken rotational symmetry
- Direction varies from point to point but magnitude is more or less fixed

Map the directions of \(M\) onto a sphere

Different points related by rotations

Vector with 3 components

Two dimensional crystal

- Important degrees of freedom associated with broken translational order
- Consider a deformation described by displacement vectors \(U(x)\)
- However which "ideal" atom is associated with which "real" atom?

\[ U(x) = U(x) + ma_i + nb_j \]

- All are equivalent points in a square
- Hence the order parameter \(U(x)\) is a square with periodic boundary conditions

Order parameter space is a torus

Excitations

- Crystals are rigid because of the broken translational symmetry
- Uniform displacements cost no energy
- Energy depends on derivatives or gradients of displacements
- Long wavelength waves have low frequencies \(\Rightarrow\) sound waves
- \(\text{broken symmetry} \Rightarrow \text{low frequency excitations (Goldstone theorem)}\)

Long wavelength modulations of the order parameter cost an energy \(\frac{1}{2}C (\frac{du}{dx})^2\)

Magnetic waves or spin waves

Topological Defects

- A defect is a tear in the order parameter field
- Consider the two dimensional crystal with an extra row of atoms called a dislocation

- Away from the middle there are small distortions
- Can we repair the defect by simple rearrangements?
- \(\text{No! the affect of the defect extends a long way}\)
- Consider a closed path surrounding the defect and count the rows crossed
- For any path enclosing the core there will be an extra row on the right

Order Parameter Space

- Moving around the loop corresponds to a loop in the order parameter space
- Deforming the atoms slightly does not change the number of times the loop winds around the hole \(\Rightarrow\) winding number describes the defect
- Loop cannot be shrunk to a point \(\Rightarrow\) topological space is not simply connected
Homotopy Classes

- If the order parameter field does not wind around the torus, it can be smoothly deformed back to a uniform state.
- If it winds around the hole in the torus or through it, then it cannot be continuously deformed back to the uniform state.
- Two integers needed to describe the defect: \( (m,n) \), Burger's vector.
- Two loops are equivalent if they can be twisted into one another to form equivalence classes.

\[ \Pi_1(T^2) = \mathbb{Z} \times \mathbb{Z} \]

**d=2 xy model**

- Kosterlitz and Thouless (1973) predicted that the transition is a defect unbinding transition.
- System acquires a rigidity at low \( T \).

Charge=0

Single vortex (charge=+1) costs infinite energy to create.

Pair of vortices (total charge zero) costs a finite amount of energy.

\[ \Pi_1(S^1) = \mathbb{Z} \]

Kosterlitz and Thouless (1973) predicted that the transition is a defect unbinding transition.

The system acquires a rigidity at low \( T \).