Charged Excitons in the Quantum Hall Regime: Optical Probe of Fractionally Charged Quasiholes

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Abstract. We review recent optical experiments on dilute two-dimensional electron systems (2DES) at very low temperatures ($T < 0.1$ K) and high magnetic fields. In photoluminescence experiments on a 2DES subjected to a quantizing magnetic field around $\nu = 1/3$, we have observed an anomalous dispersion of the charged excitons. We have found that the anomaly exists only at a very low temperature (0.1 K) and an intermediate electron density (0.9×10^{11} cm^{-2}). It is explained to occur due to the perturbation of the incompressible liquid at $\nu = 1/3$. The perturbation is induced by the close proximity of a localized charged exciton which creates a fractionally-charged quasihole in the liquid. The intriguing experimentally observed puzzle that the anomaly (2 meV) can be destroyed by applying a small thermal energy of ~ 0.2 meV is thereby resolved, as this energy is enough to close the quasihole energy gap. This work presents a first ever probe of the quasihole gap in a quantum Hall system.

1 Introduction

Since its discovery [1], the fractional quantum Hall effect (FQHE), which is a direct manifestation of many-particle interactions in a two-dimensional electron system (2DES), has attracted great and sustaining research interest. The past decades have shown that, in addition to transport experiments, optical experiments like photoluminescence (PL) spectroscopy can provide further and in some sense complementary information about the ground state of quantum Hall systems. In fact, in optical experiments one is able to probe

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the bulk of a quantum Hall system, whereas transport is mostly governed by edge channels. However, one of the striking differences of optical experiments, compared to transport, is the presence of photoexcited holes in the system, which can significantly influence its properties.

The optical investigations reported so far can roughly be devided into four categories by the systems under investigation: (i) single heterojunctions (e.g., Ref. [2]) (ii) remotely acceptor-doped heterojunctions (e.g, Ref. [3,4]), and (iii) single quantum wells (e.g., Ref. [5,6,7,8,9,10]). In particular in experiments on quantum wells, which are also the topic of this work, electrons and photoexcited holes reside essentially in the same spatial layer – the quantum well – and Coulomb interaction will be of major importance. Though it was previously shown that in symmetric systems in the high magnetic field limit the electron-electron and electron-hole interactions cancel exactly due to a hidden symmetry [11,12,13], for most of the real systems this symmetry is violated. The role of Coulomb interaction between electrons and holes in optical experiments has been explored in recent years, in particular, in experiments on very dilute 2DES, in the density range $\sim 10^{10}\,\mathrm{cm}^{-2}$. Here, it was found that the ground state is formed by negatively charged excitons, a bound state of two electrons and one hole. In a magnetic field, the two electrons can form either a singlet or a triplet state; the excitons are then called singlet (X_s^-) or triplet (X_t^-) excitons, respectively (see, e.g., [14,15,16,17,18,19,20,21,22]). For samples with relatively strong disorder, the commonly accepted picture is that at low density the 2DES breaks up into areas with finite density (electron puddles) and completely depleted regions, where neutral excitons (X^0) can form. On the other hand, experiments on high-quality GaAs-AlGaAs samples suggested [23] that charged and neutral excitons may also reside in the same spatial regions. This is, however, controversially discussed, since, theoretically, neutral excitons should not be stable if charged excitons exist as bound states at lower energies. Furthermore, time-resolved experiments on p-doped CdTe-CdMgZnTe quantum wells [24] revealed that, both the positively charged excitons and the residual holes are localized, whereas the neutral excitons remain freely moving. We will show below that the effect of disorder is one of the main ingredients in our experiments, too.

At higher densities, in the range $\sim 10^{11}$ cm^{-2}, the Coulomb interaction between electrons and holes is mostly screened. Naturally, the crossover between these two regimes should reveal interesting interaction related effects. Indeed, in our work we could show that the charged excitons can, under specific conditions, be used as a tool to investigate the properties of a highly correlated quantum Hall liquid [10]. We will show that the interaction between a localized charged exciton and the incompressible liquid in the $\nu = 1/3$ state leads to the creation of a fractionally-charged hole in the liquid. This low-lying charged excitation is a fundamental consequence of the incompressible nature of the ground state – the liquid state proposed by Laughlin [25].

In addition to the ground state energy, the quasihole creation energy for the state proposed by Laughlin is also known very accurately [26]. The energy gap for a well-separated quasiparticle – quasihole pair has been derived from low-temperature measurements [27], and theoretically investigated by many authors [28,29]. A few years ago, several groups reported their successful attempts to detect the existence of fractionally charged objects in the edge states of a two-dimensional electron system invoking a chiral Luttinger liquid theory [30]. In our work, we have demonstrated that the lowest-energy charged excitations, the quasiholes, can be probed in an incompressible liquid at $\nu = 1/3$ (the Laughlin state) via optical spectroscopy [10].

2 Magneto-photoluminescence and Absorption in the Quantum Hall Regime

2.1 Experimental Details

The samples which were investigated in this work are one-sided modulation-doped GaAs-$Al_{0.33}Ga_{0.67}As$ single quantum wells with a 25 nm wide GaAs quantum well where a 2DES is formed. One one side of the quantum well, either a 70 nm or a 88 nm AlGaAs spacer separates the doped barrier region from the quantum well. On the other side, a 100 period AlGaAs/GaAs (10 nm/3 nm per period) superlattice was grown. On top of the sample, a 7 nm thick Titanium gate was deposited. The sample was glued upside down on a glass substrate using an UV curing optical adhesive. Subsequently, the sample was thinned from the back side by a selective etching process [31] down to the superlattice to a total thickness of about 1.3 μm. By applying a negative gate voltage, we can tune the density in the range between about 1×10^{10} cm^{-2} and 2×10^{11} cm^{-2}. PL and absorption measurements were performed either in an optical split-coil cryostat, or via glass fibers in a ^{3}He/^{4}He dilution cryostat at a base temperature of $T = 40$ mK and magnetic fields up to 16 T. In the dilution cryostat a sensor at the sample position indicated that during illumination the temperature directly at the sample is about $T = 100$ mK, while the base temperature is still $T = 40$ mK. Circularly polarized light was created directly inside the mixing chamber and left and right circularly polarized spectra were measured by ramping the magnet from positive to negative magnetic fields. For the PL, the sample was excited by a Ti:Sapphire laser at 750 nm. For the absorption measurement, a white light source was used.

2.2 High Electron Density

If a perpendicular magnetic field is applied to the 2DES, the constant density of states condenses into discrete Landau levels (see Fig. 1a). Each Landau level exhibits a Zeeman splitting (omitted in Fig. 1a), $g\mu_B B$. The g-factor of

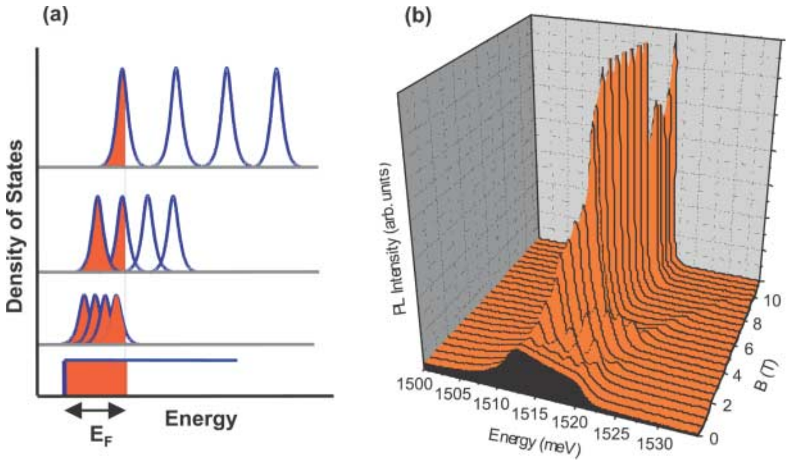

Fig. 1. (a) Schematic picture of the density of states of a 2DES for increasing magnetic field (from bottom to top panels). Gray-shaded areas should indicate occupied states. (b) Experimental PL spectra of a sample with carrier density $n = 2.3 \times 10^{11}$ cm^{-2} at $T = 2$ K for different magnetic fields

electrons in GaAs, $g = -0.44$, arises from band-structure effects (μ_B is the Bohr magneton), while additional splittings arise from the strong Coulomb correlations. Each spin-split Landau level has a degeneracy, which is determined by the number of magnetic flux quanta per area, eB/h. The filling factor ν of the system is defined as the number of electrons in the 2DES devided by the number of flux quanta, $\nu = nh/(eB)$, where n is the density of electrons in the 2DES. This means that if ν is an integer, we have ν completely filled spin-split Landau levels. At these integer values, plateaus are observed in a Hall measurement. Figure 1b displays a series of experimental PL spectra for different magnetic fields of a sample with a carrier density of $n = 2.3 \times 10^{11}$ cm^{-2}. At this relatively large density, the Coulomb interaction between electrons and photoexcited holes is mostly screened and one observes PL lines which can be interpreted as recombination of electrons from the occupied Landau levels in the conduction band with photoexcited holes in the valence band. In the experimental spectra in Fig. 1b, one can see the evolution of occupied Landau levels with magnetic field. At integer ν the Fermi energy jumps from one Level to the next lower level, and the PL from the higher level, which is empty now, decreases abruptly. This can especially be seen at $B = 4.5$ T in Fig. 1b where $\nu = 2$. The weak signal from the third Landau level for magnetic fields larger than 4.5 T in Fig. 1b stems from photoexcited e-h pairs. At the same time, at integer ν, there are pronounced anomalies in the intensity and energetic position of the recombination from the lowest Landau level. These can for instance be seen at $\nu = 2$ ($B = 4.5$ T in Fig. 1b) and $\nu = 1$ ($B = 9$ T in Fig. 1b) where the 2DES is completely spin polarized. There is a variety of literature on these many-particle interaction

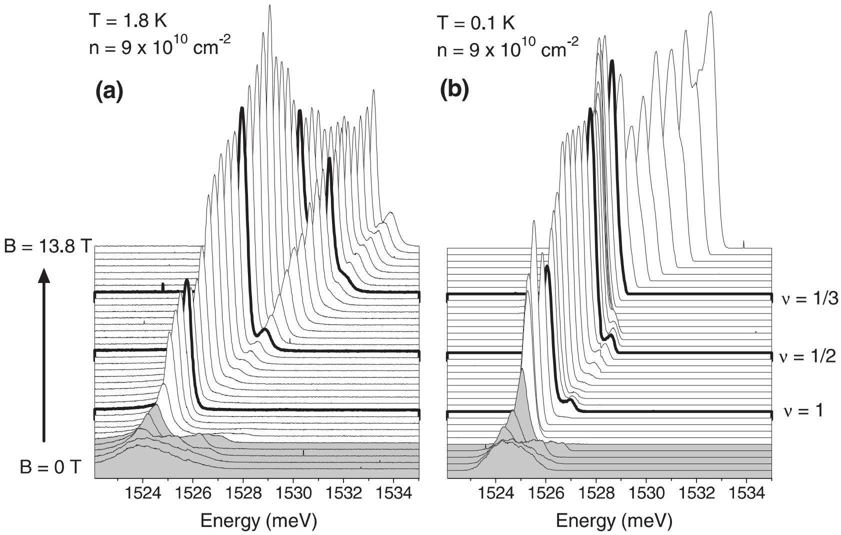

Fig. 2. PL spectra for different magnetic fields for a carrier density of $\sim 0.9 \times 10^{11}\,\mathrm{cm}^{-2}$ at a temperature of (a) $T = 1.8$ K and (b) $T = 0.1$ K

induced effects in PL in high-density samples (see, e.g., [2,5,6]), which shall not be the focus of this work.

2.3 Intermediate and Low Electron Density

For samples with moderate electron mobility, in the range of $10^5\,\mathrm{cm}^2/\mathrm{Vs}$, we expect that for densities below about $1 \times 10^{11}\,\mathrm{cm}^{-2}$ the residual disorder plays a significant role and localization effects of charged excitons might be of importance. In the following we will call this density regime, where we think that in our samples a uniform 2DES just starts to form, the intermediate density regime. Figure 2 shows PL spectra for an electron density of $0.9 \times 10^{11}\,\mathrm{cm}^{-2}$ at a temperature of $T = 1.8\,\mathrm{K}$ (Fig. 2a) and $T = 0.1\,\mathrm{K}$ (Fig. 2b). Note that the only difference between the experiments displayed in Figs. 2a and 2b is the different temperature. For filling factors $\nu > 2$, i.e., if the lowest Landau level is completely occupied, we find qualitatively the same behavior in both experiments (gray shaded spectra in Figs. 2a and 2b): The observed PL lines show a nearly linear magnetic-field dispersion as observed in the high-density case (Fig. 1a). This can be attributed to recombinations of electrons from the occupied Landau levels with photocreated holes in the valence band. At $\nu = 2$, we find an abrupt crossover to a regime where negatively charged singlet excitons form, which we infer from the nearly quadratic dispersion of the observed PL line for $\nu < 2$. A similar behavior, i.e., a crossover between a Landau level-like and a charged exciton regime at $\nu = 2$, was reported recently [23,32] for symmetric quantum wells,

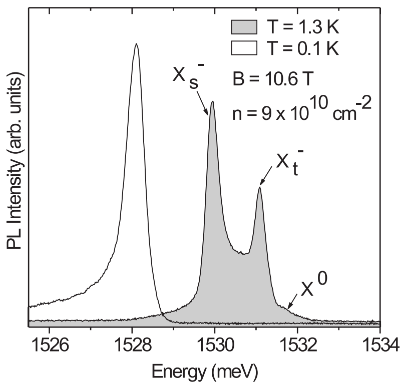

Fig. 3. Left-circularly polarized photoluminescence spectra for a carrier density of $9 \times 10^{10}\,\mathrm{cm}^{-2}$ and a magnetic field of B=10.6 T, corresponding to $\nu = 1/3$, at temperatures 0.1 K and 1.3 K

and was attributed to a breaking of the so called hidden symmetry [11,12] for $\nu > 2$ [32]. We note that we observe here a similar behavior in *asymmetric* quantum wells.

The interesting region, however, where we observe significant differences in the experiments displayed in Fig. 2, is the magnetic field range where $\nu < 1$. Here, the measurements at $T = 1.8$ K (Fig. 2a) exhibit the well known formation of negatively charged excitons: The excitation with lowest energy, the X_{s}^-, is visible over the whole magnetic field range for $\nu < 2$. For $\nu < 1$, the bright triplet exciton X_{tb}^- [8,33] gradually appears, and, at high fields, the neutral exciton X^0 shows up at the high energy flank of the X_{tb}^-. The situation changes drastically if we lower the temperature to 0.1 K (Fig. 2b). Here, the X_{tb}^- appears abruptly at $\nu = 1$, and, more strikingly, the lowest energy line, presumably the X_{s}^-, shows a strong down curvature in energy for filling factors $\nu < 1/2$. To emphasize this experimental finding, in Fig. 3 two spectra, taken at $B = 10.6$ T, which roughly corresponds to filling factor $\nu = 1/3$, are compared. Note that, again, the only difference between the two spectra is the different temperature. The spectrum at $T = 1.3$ K shows the charged and neutral excitons, whereas at $T = 0.1$ K only a single line, which is strongly redshifted, is observed. To analyze this behavior in more detail, in Fig. 4 the energetic positions of the observed lines at $T = 1.8$ K (open symbols) and $T = 0.1$ K (solid symbols) are plotted versus magnetic field for a density of 0.9×10^{11} cm^{-2}. One can see that, starting at $\nu = 1$, the energies of both, X_{s}^- (solid circles in Fig. 4) and X_{tb}^- (solid uptriangles in Fig. 4), are lowered in the experiment at $T = 0.1$ K with respect to the experiment at $T = 1.3$ K (open symbols in Fig. 4). The inset of Fig. 4 shows the experimentally determined anomaly ΔE versus magnetic field. The most striking result here is that a thermal energy of ~ 2 K (~ 0.2 meV) is sufficient to completely destroy the anomaly which has a strength of about 2 meV. This rules out any trivial localization effect of excitons, since then one would expect that at least a thermal energy of also about 2 meV would be necessary to delocalize the excitons.

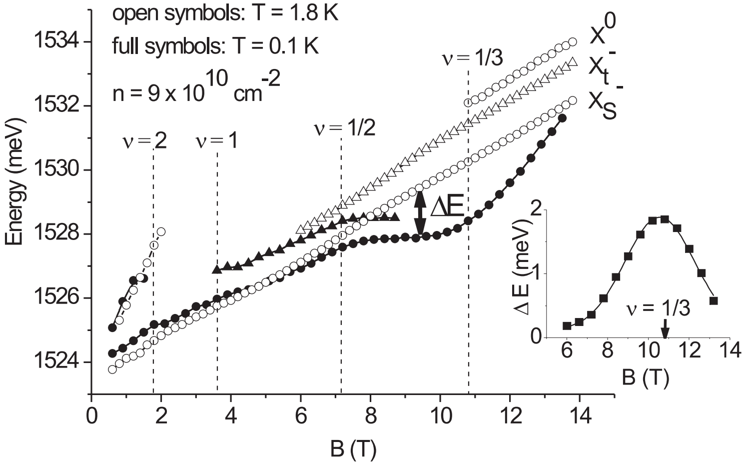

Fig. 4. Experimentally observed mode positions at $T = 1.8$ K (open symbols) and $T = 0.1$ K (solid symbols). The inset shows the anomaly ΔE versus magnetic field

To get more information about the possible origin of the anomaly, we have also performed direct absorption measurements at low temperatures. Absorption experiments give direct information about the oscillator strengths of the observed excitations, while the intensities of PL lines can also be dominated by effects like localization of excitons at impurities or potential fluctuations. Figure 5a shows a series of absorption spectra using white light for different magnetic fields. The displayed spectra $S(E)$ are calculated from the measured transmission spectra $T(E)$ by the relation $S(E) = -ln(T(E)/T_0(E))$, where $T_0(E)$ is the measured transmission without sample. Due to the relatively strong illumination with white light, also PL recombinations are initiated and add to the transmission signals. Therefore, in the normalized spectra in Fig. 5a, PL lines appear as dips, while absorption lines appear as peaks. Figure 5b shows the extracted dispersions of the PL and the absorption lines which we attribute to the X_{s}^- and X_{tb}^- excitons. The interesting result is that the absorption shows the same anomaly as the PL line, which is a strong indication that the anomaly is an intrinsic effect and is not caused by localization of excitons. In the case of localization, one would expect that the absorption should still reflect dominantly the intrinsic structure of the system, whereas the PL might be dominated by extrinsic effects, like exciton localization.

Interestingly, if we lower the density to the range $\sim 10^{10}$ cm^{-2}, the anomaly disappears completely, even in experiments at very low temperature. This can be seen in Fig. 6, where contour plots of PL spectra are shown for three different densities. One can see the well-known behavior of the charged excitons, and, for filling factors $\nu < 1/3$, the so called dark triplet exciton X_{td}^-, which has been observed in experiments at very low temperature [7,8]. This charged triplet exciton was found to exhibit the singlet-triplet crossing

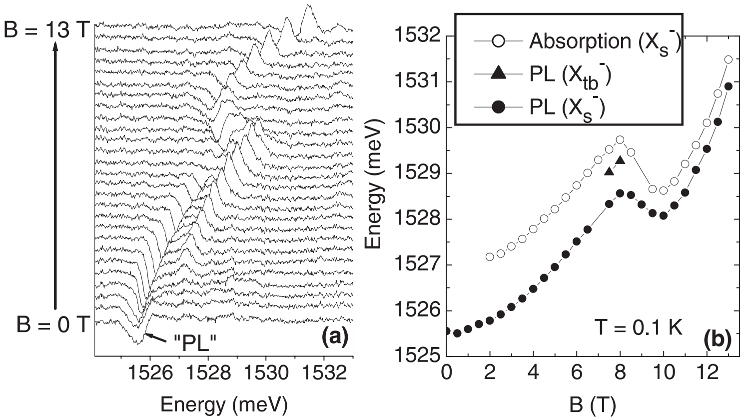

Fig. 5. (a) Normalized (see text) transmission spectra (unpolarized) for different magnetic fields. (b) Energetic positions of PL and absorption lines versus magnetic field

at high magnetic fields [33,34]. However, the important information of Fig. 6 for the investigation here is that in the range of filling factor $\nu = 1/3$ no anomalous dispersion of the charged excitons can be observed.

Before we give an explaination for the anomaly around $\nu = 1/3$, we want to summarize at this point the relevant facts from the experiments: (a) The anomaly is not seen at *higher* electron densities where no charged excitons but usual electrons exist, and is also not seen for *lower* electron densities where exclusively charged excitons are present [8]. Also for higher mobility samples ($\sim 5 \times 10^6$ cm^2/Vs, not shown here) it is not present. Because of the low mobility and relatively low density of the sample, excitons are expected to remain localized. (b) The anomaly appears near 1/3, i.e., excitons are near an incompressible liquid. (c) The most intriguing observation is that *a very small thermal energy* ($\ll 2$ *meV) is required to destroy the anomaly.* (d) The anomaly does not appear near $\nu = 1, 2$ and is therefore an indication that the lowest-energy charged excitations, the quasiholes (for reasons to be discussed below) are perhaps involved in the process. The quasielectrons are predicted to have higher energies [28].

3 Theoretical Model: Creation of Fractionally-Charged Quasiholes

In our explanation of the observed anomaly [10] we assume that, as a result of potential fluctuations due to impurities in the system, excitons remain localized but they are in close proximity to the incompressible liquid at $\nu = 1/3$. Two of us recently investigated a system [35] where a parabolic quantum dot (QD) [36] is coupled (via the Coulomb force) to a 2DES which is in a

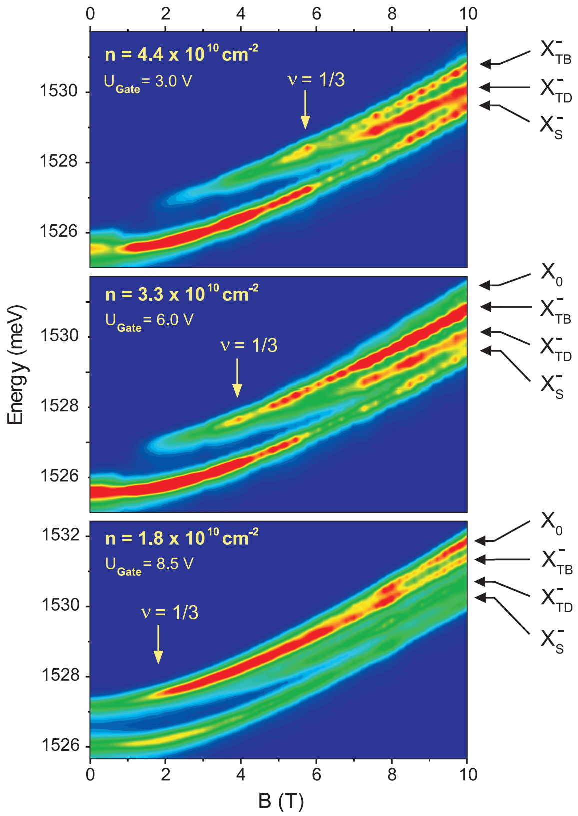

Fig. 6. Contour plots of PL experiments at T = 0.1 K for different carrier densities

$\nu = 1/3$ Laughlin state. Electrons in the dot are confined by a parabolic potential [36], $V_{\rm conf}(x,y) = \frac{1}{2}m^*\omega_0^2\left(x^2+y^2\right)$, where ω_0 is the confinement potential strength and the corresponding oscillator length is $l_{\rm dot} = (\hbar/m^*\omega_0)^{1/2}$. Calculating the low-energy excitations of that quantum dot-liquid system (named a *qd-liquid*) we found that for a single electron in the dot the physics is somewhat similar to that of a point impurity in a $\nu = \frac{1}{3}$ liquid state investigated earlier [37]. In this case, the QD emits a fractionally-charged *quasihole* $(e/3)$ that orbits around the QD, as evidenced from the charge-density calculations [35,37]. Here we propose that the observed anomaly is related to the qd-liquid where the QD contains a charged exciton. The QD in our model of Ref. [35] represents a localized exciton (charged or neutral) in the present case, and perturbs the incompressible fluid due to its close proximity by creating fractionally-charged defects. Details on the formal aspects of our theory can be found in Ref. [35]. We model the incompressible state at $\nu = 1/3$ filling using the spherical geometry [29] for six electrons. Electrons are treated as spinless particles corresponding to the state described by the Laughlin wavefunction [25]. We consider the QD size $l_{\rm dot} = 15$ nm and the

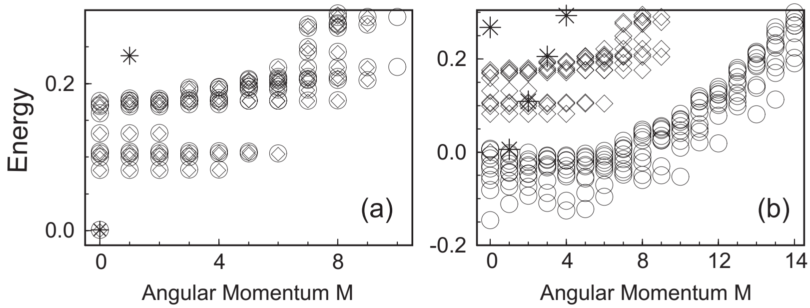

Fig. 7. Energy (in units of Coulomb energy) versus the azimuthal rotational quantum number M for an isolated quantum dot ($*$), a two-dimensional electron liquid ($\diamond$), and a qd-liquid ($\circ$). The QD of the qd-liquid either contains $(1e, h)$ [in (a)], or $(2e, h)$ [in (b)]

liquid-dot separation $d = 1.5l_0$. The QD contains either a pair of electron and hole (e, h) (charge-neutral QD), or $(2e, h)$ (charged QD). Figure 7 shows the energy spectra for the qd-liquid where the QD contains either (e, h) [in (a)] or $(2e, h)$ [in (b)]. In the figures, the energy spectra of isolated dots ($*$), an incompressible liquid at $\nu = 1/3$ state ($\diamond$) and the binding energy of the QD to the incompressible liquid ($\circ$) are plotted for comparison. From Fig. 7a we can see that for a charge-neutral dot there is no dispersion of the energy as a function of M, and most importantly, the incompressible liquid is not influenced by the dot at all. On the other hand, the energy of the qd-liquid is significantly lowered for a charged QD, as compared to the isolated QD or the incompressible liquid without the dot [Fig. 7b]. This is in line with the experimental observation where only the charged excitons show the anomaly by lowering the energy.

Figure 8 exhibits the electron density distribution in the liquid (L) and in the dot (QD) for the lowest states and for a given angular momentum of the qd-liquid system. The electron (or hole) system in the dot is close to the ground state of an isolated dot, i.e., the influence of the incompressible liquid on it is very small. On the other hand, the low-lying excited states of the qd-liquid can be described by the ionization process [35,37] as emission of a quasihole: Since the net charge of a QD is negative, the ground state of the qd-liquid can be considered as a QD plus three quasiholes. If we increase the angular momentum of the 2D electrons, one of the quasiholes moves away from the QD. This is inferred from the calculated charge distribution of electrons around the QD, where a local minimum corresponds to the quasihole moving away from the QD as the angular momentum is increased. The position of the local minimum at different angular momenta of the charge density correspond to the orbit radius of the quasihole [35,37]. We have evaluated the quasihole creation energy in a qd-liquid. For the Laughlin state in a pure 2DES it is 0.0276 $e^2/\epsilon l_0$ [26]. For the qd-liquid, the corresponding

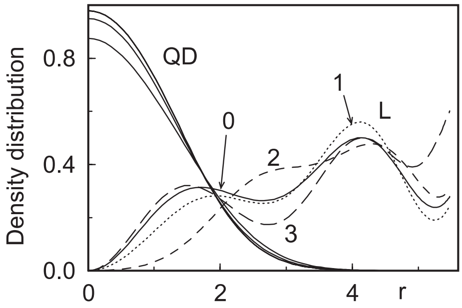

Fig. 8. Charge-density profile of the ground state and low-lying excitations of electrons in the dot (QD) and the liquid (L) of a qd-liquid at $\nu = 1/3$ (for $M = 0 - 3$, as indicated in the figure) corresponding to Fig. 7b

value is 0.32 meV and is expected to decrease a little further with increasing number of electrons in the system representing the incompressible liquid [29]. This result indicates that the small thermal energy of about 0.2 meV required to destroy the anomaly is in fact, the quasihole energy gap.

4 Conclusion

PL experiments on a 2DES subjected to a quantizing magnetic field exhibit a significant lowering of exciton energies at and around $\nu = 1/3$. This anomalous dispersion is explained as due to the perturbation of the incompressible liquid at $\nu = 1/3$ by localized charged excitons which results in the creation of fractionally-charged quasiholes in the liquid.

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