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Incompressible states of dirac fermions in graphene with anisotropic interactions

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ABSTRACT

We report on the properties of incompressible states of Dirac fermions in graphene in the presence of an anisotropic Hamiltonian and a quantizing magnetic field. We introduce the necessary formalism to incorporate the unimodular spatial metric to deal with the anisotropy in the system. The incompressible state in graphene is found to survive the anisotropy up to a critical value of the anisotropy parameter. The anisotropy also introduces two branches in the collective excitations of the corresponding Laughlin state. It strongly influences the short-range behavior of the pair-correlation functions in the incompressible ground state.

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1. Introduction

In his quest for a better understanding of the Laughlin state [1], which is widely regarded as the best description of the fractional quantum Hall effect (FQHE) ground states [2] at the primary filling fractions ($\nu = 1/m$), Haldane [3] recently demonstrated that the integer and the fractional quantum Hall effects are fundamentally different. In the latter case, he introduced a unimodular (area preserving) spatial metric that characterizes the shape of the correlation functions of the Laughlin state and is obtained by minimizing the correlation energy of the fractional quantum Hall state. This interaction metric is not necessarily congruent to the Galilean metric present in the one-body term of the system Hamiltonian. Such a geometrical degree of freedom of the Hamiltonian is totally absent in the integer quantum Hall effect, but its presence helps to explain the success of the many-body state of Laughlin. Subsequent numerical studies [4–6] have elucidated various properties of the incompressible FQHE states in the presence of anisotropic interactions. Anisotropic transport in the FQHE regime exists in higher order filling fractions [7], and has also received some theoretical attention [8]. It is naturally interesting to examine the role of unimodular spatial metrics in the incompressible state of massless (no Galilean metric) Dirac fermions.

Of late, there has been an upsurge of interest on the magnetic field effects on Dirac fermions in graphene [9]. The FQHE states

were investigated by us in monolayer [10] and bilayer [11] graphene. Experimentally, the presence of this effect in graphene has been confirmed [12,13]. Interactions among Dirac fermions play an important role in the quantum Hall regime of graphene [14]. Here we investigate the nature of the incompressible state of graphene with anisotropic interactions, which can be introduced through anisotropic dielectric properties of a substrate. Anisotropic interactions result in several unique features of the FQHE state, e.g., in the collective modes and in the pair-correlation functions.

2. Model and main equations

Monolayer graphene in a magnetic field B has a discrete Landau level (LL) energy spectrum that is characterized by the LL index $n = 0, \pm 1, \pm 2, \dots$ and energy [9,14]

$$\varepsilon_n = \hbar\omega_B \operatorname{sgn}(n)\sqrt{|n|}, \quad (1)$$

where $\omega_B = \sqrt{2}v_F/\ell_0$ and $\ell_0 = \sqrt{\hbar/eB}$ is the magnetic length. Here $\operatorname{sgn}(n) = 0$ if $n = 0$ and $\operatorname{sgn}(n) = \pm 1$ if $n > 0$ and $n < 0$, respectively. The corresponding wavefunctions are

$$\Psi_{n,m} = C_n \begin{pmatrix} \operatorname{sgn}(n)i^{|n|-1}\phi_{|n|-1,m} \\ i^{|n|}\phi_{|n|,m} \end{pmatrix}, \quad (2)$$

where $C_{n=0} = 1$ and $C_{n \neq 0} = 1/\sqrt{2}$. The functions $\phi_{n,m}$ are the conventional Landau wavefunctions with the LL index n and the z -component of electron angular momentum m . These functions are characterized by two sets of ladder operators: operator b^\dagger , which raises the LL index n , and the guiding center ladder operator

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a^\dagger , which raises the intra-Landau index m . The energy spectrum depends only on n and is highly degenerate with respect to the electron angular momentum m . The LL wavefunctions in Eq. (2) are isotropic, i.e., the electron density depends only on $\rho = \sqrt{x^2 + y^2}$. The basis functions (2) are however, not unique. Due to degeneracy of the LLs we can choose any single-particle basis, even anisotropic ones, to describe the properties of the electron system in a strong magnetic field. For the many-electron system with isotropic interactions the wavefunctions (2) are the convenient basis for evaluating the many-electron energy spectrum in a spherical geometry, since the isotropic potential conserves the angular momentum. In spherical geometry [15], the interaction properties of the many-electron system are described in terms of the Haldane pseudopotentials V_m [15], which are the energies of two electrons with relative angular momentum m . The radius of the sphere, R , is related to the integer $2S$ of magnetic fluxes through the sphere in units of the flux quanta, $R = \sqrt{S}\ell_0$. The single-electron states are characterized by the angular momentum, which is equal to S , and $S_z = -S, \dots, S$. For an isotropic potential the many-particle states are described by the total angular momentum, L , and L_z , while the energy depends only on L . We can then evaluate the energy spectra of the system for a given value of L_z , e.g., $L_z = 0$ [16], which greatly simplifies our analysis of the many-electron system in a given LL.

We consider the anisotropic electron–electron interaction potential of the type

$$V(\mathbf{r}_1, \mathbf{r}_2) = \frac{e^2}{\kappa \sqrt{(x_2 - x_1)^2(1 + \gamma) + (y_2 - y_1)^2(1 - \gamma)}}, \quad (3)$$

where $\mathbf{r}_1 = (x_1, y_1)$ and $\mathbf{r}_2 = (x_2, y_2)$ are the coordinates of two electrons, κ is the dielectric constant, and the parameter γ characterizes the anisotropy of the interaction potential with $\gamma = 0$ corresponding to isotropic interaction. We study the incompressible states in graphene with fractional filling factor and with the anisotropic interaction (3). We assume that the interaction does not mix the states of different LLs. In this case the interaction potential should be projected on a given LL with index n . Although the interaction potential within a given LL is anisotropic, it can be made isotropic by a non-uniform scaling transformation of the coordinates

$$x' = x\sqrt{1 + \gamma}; \quad y' = y\sqrt{1 - \gamma}. \quad (4)$$

The interaction then becomes isotropic, $V \propto 1/|\mathbf{r}'_1 - \mathbf{r}'_2|^2$. In the scaled coordinate system the Hamiltonian of a single electron in graphene becomes to the anisotropic, which corresponds to anisotropic Fermi velocity.

To take advantage of the isotropic potential in the scaled coordinate system we need to define the angular momentum which is conserved by the isotropic interaction. Therefore we choose a single-electron basis within a given LL, which is initially anisotropic, and define the angular momentum in that anisotropic basis. This procedure was followed in Ref. [4], where the anisotropic LL basis states with the non-Euclidean guiding center metrics were introduced for the conventional LLs. These metrics are characterized by the anisotropy parameter γ . The corresponding guiding center intra-LL ladder operators are introduced through the Bogoliubov transformation $a_\gamma = (1/\sqrt{1 - \gamma^2})(a + \gamma a^\dagger)$; $a_\gamma^\dagger = (1/\sqrt{1 - \gamma^2})(\gamma a + a^\dagger)$. The LL basis states of the non-relativistic system with parabolic dispersion relation are then

$$\phi_{n,m}(\gamma) = \frac{(b^\dagger)^n (a_\gamma^\dagger)^m}{\sqrt{n!m!}} \phi_{0,0}(\gamma), \quad (5)$$

where the state $\phi_{0,0}(\gamma)$ is determined by the condition $a_\gamma \phi_{0,0}(\gamma) = b \phi_{0,0}(\gamma) = 0$.

In the coordinate representation the zeroth conventional LL function $\phi_{0,0}(\gamma)$ of nonrelativistic electrons is

$$\phi_{0,0}(\gamma) = \frac{(1 - \gamma^2)^{1/4}}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\gamma z^2 - \frac{1}{2}|z|^2\right), \quad (6)$$

where $z = x + iy$ is the complex coordinate. Expressing the ladder operators in terms of the complex coordinate, $a = \frac{1}{2}z^* + \partial_z$ and $b = \frac{1}{2}z + \partial_{z^*}$, and using the expression for the anisotropic ladder operators a_γ , the anisotropic LL basis states are constructed as

$$\Psi_{n,m}(\gamma) = C_n \begin{pmatrix} \text{sgn}(n) i^{|n|-1} \phi_{|n|-1,m}(\gamma) \\ i^{|n|} \phi_{|n|,m}(\gamma) \end{pmatrix}. \quad (7)$$

The angular momentum in this anisotropic basis is m and $L_z(\gamma) = a^\dagger a$. For the many-electron system with anisotropic interaction and anisotropic basis states, the interaction properties are characterized by the pseudopotentials, V_m , with relative angular momentum $m = m_1 - m_2$. Therefore the interaction energy of two electrons depends only on their relative momentum, but not on the total momentum, $M = m_1 + m_2$. The many-electron system with anisotropic interaction can then be studied in a spherical geometry, where the total angular momentum is conserved.

We evaluate V_m in the planar geometry with $\Psi_{n,m}(\gamma)$ and use these values in the spherical geometry to find the energy spectra of the many-electron system. The pseudopotentials V_m are obtained from

$$V_m = \int d\mathbf{r}_1 d\mathbf{r}_2 |\Phi_m(\gamma, \mathbf{r}_1, \mathbf{r}_2)|^2 V(\mathbf{r}_1, \mathbf{r}_2), \quad (8)$$

where the two-electron state $\Phi_m(\gamma, \mathbf{r}_1, \mathbf{r}_2)$ with relative angular momentum m and $M=0$ is

$$\Phi_m(\gamma, \mathbf{r}_1, \mathbf{r}_2) = \frac{(a_{\gamma,1}^\dagger - a_{\gamma,2}^\dagger)^m}{\sqrt{2^m m!}} \Psi_{n,0}(\gamma, \mathbf{r}_1) \Psi_{n,0}(\gamma, \mathbf{r}_2). \quad (9)$$

Here $a_{\gamma,1}^\dagger$ and $a_{\gamma,2}^\dagger$ are the guiding center ladder operators for electrons 1 and 2, respectively.

3. Results and discussion

In graphene, the FQHE with a large many-particle excitation gap occurs only for $n=0$ and $n=1$, where the FQHE gap is the largest for the $n=1$ LL [10,14]. Here we consider only those two LLs. To characterize the interaction properties of a partially occupied LLs, we study the FQHE for a filling factor $\nu = 1/3$. Similar behavior is expected for other filling factors, e.g., $\nu = 1/5, 2/3$. The wavefunction of the $n=0$ graphene LL is identical to that of the $n=0$ conventional LL of the nonrelativistic system. Therefore in this case the interaction properties and the pseudopotentials of graphene and conventional (non-relativistic) systems are identical. The $n=1$ graphene LL wavefunction, on the other hand, is a mixture of $n=0$ and $n=1$ conventional Landau functions that is responsible for several unique properties of graphene [14].

The magnitude of the FQHE gap depends on how V_m decreases with m . The FQHE with a large gap is characterized by the large values of the ratios V_1/V_3 and V_3/V_5 . In Fig. 1 we show the ratio of the pseudopotentials as a function of γ . For all values of γ , the ratios are the largest for $n=1$, which suggest that the FQHE state is more stable in the $n=1$ LL. For increasing γ , the ratios decrease for both $n=0$ and $n=1$, which makes the FQHE less stable for large γ . These results suggest that the gap decreases with increasing γ and finally collapses for $\gamma \approx 0.8$, when the ratios are close to one. To find the dependence of the FQHE gap on the anisotropy, we evaluate the energy spectra and the excitation gap of a finite-size system comprising $N=8$ electrons. The $\nu = \frac{1}{3}$ gap is shown in Fig. 2 as

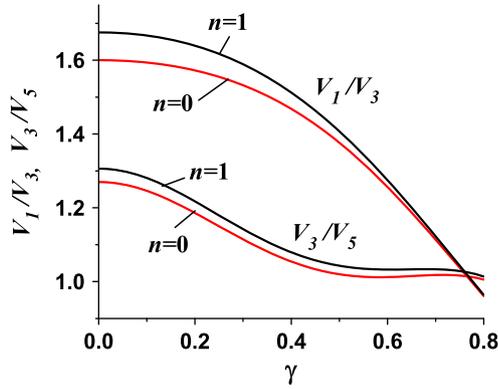


Fig. 1. (Color online) The ratios of the pseudopotentials, V_1/V_3 and V_3/V_5 as a function of γ in two graphene LLs $n=0$ (red lines) and $n=1$ (black lines).

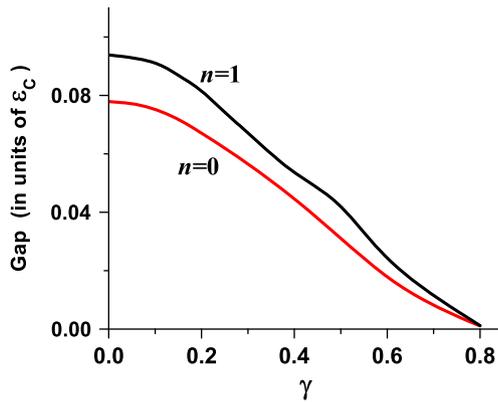


Fig. 2. (Color online) The $\nu = \frac{1}{3}$ -FQHE gap as a function of γ for the eight-electron system and $n=0$ (red line) and $n=1$ (black line). The gap is evaluated in the spherical geometry with flux quanta $2S=21$.

a function of γ . For increasing γ the gap decreases in both LLs, with the gap being larger for $n=1$. For $\gamma = \gamma_{cr} \approx 0.8$ the gap disappears and the $\nu = \frac{1}{3}$ state becomes compressible. The critical value γ_{cr} is the same for both LLs. This value corresponds to the condition that the pseudopotentials V_1 , V_3 , and V_5 become almost the same (Fig. 1). For $\gamma < 0.15$ the gap shows weak dependence on γ .

In spherical geometry the energy dispersion is obtained as a function of L . Each state has a $(2L+1)$ degeneracy. The typical energy spectrum is shown in Fig. 2(a) for $\gamma=0.4$. The energy spectrum has a finite gap and the low energy excited states of the spectrum has well defined single energy branch, showing a roton minimum at finite value of L ($L=5$ in Fig. 2(a)). In spherical geometry, the excited energy branch is described as a function of the angular momentum $E = E(L)$. Transition to the planar geometry is realized by replacing the angular momentum L by the magnitude of the wavevector $k = L/R$, where R is the radius of the sphere. In this case the energy of the lowest excited states depends on the magnitude of the wavevector but not on its direction, which corresponds to an isotropic system.

For anisotropic interactions with anisotropic basis, the energy depends not only on k but also on its direction. Such an anisotropic system can be made isotropic under a scaling coordinate transformation determined by Eq. (4). The corresponding transformation in the wavevector space is

$$k'_x = k_x / \sqrt{1+\gamma}; \quad k'_y = k_y / \sqrt{1-\gamma}. \quad (10)$$

Each state in the spherical geometry has $(2L+1)$ degeneracy. In the planar geometry these states correspond to $(2L+1)$ different

directions of the wavevector, $\beta_p = 2\pi p / (2L+1)$, where $p = 1, \dots, 2L+1$. The corresponding components of the wavevector are

$$k'_x = \frac{L}{R} \cos \phi_n = \frac{L}{R} \cos \frac{2\pi p}{2L+1}, \quad (11)$$

$$k'_y = \frac{L}{R} \sin \phi_n = \frac{L}{R} \sin \frac{2\pi p}{2L+1}. \quad (12)$$

These are the components of the wavevector in the scaling coordinate system [Eqs. (10)]. In this case the magnitude of the wavevector depends only on L : $(k'_x)^2 + (k'_y)^2 = (L/R)^2$ and not on the direction.

In the original coordinate system the components of the wavevector are

$$k_x = \sqrt{1+\gamma} k'_x = \sqrt{1+\gamma} \frac{L}{R} \cos \frac{2\pi p}{2L+1}, \quad (13)$$

$$k_y = \sqrt{1-\gamma} k'_y = \sqrt{1-\gamma} \frac{L}{R} \sin \frac{2\pi p}{2L+1}. \quad (14)$$

Then the magnitude of the wavevector is

$$k_p = \sqrt{k_x^2 + k_y^2} = \frac{L}{R} \sqrt{1+\gamma} \cos \frac{4\pi p}{2L+1}, \quad (15)$$

which now depends on the direction. In the original coordinate system each energy level $E(L)$ with a given angular momentum L generates $(2L+1)$ states with different wavevectors k_p . In the thermodynamic limit ($R \rightarrow \infty$), these wavevectors accumulate in two directions corresponding to the points of large density, which is proportional to $1/(dk_p/dp)$. This condition determines two values of the wavevector, $k_1 = L/R\sqrt{1+\gamma}$ and $k_2 = L/R\sqrt{1-\gamma}$. The corresponding two branches in the low-energy dispersion relation of graphene are shown in Fig. 3(b) for $\gamma=0.4$. The energy dispersion shown in Fig. 3(b) is recalculated from the energy spectrum

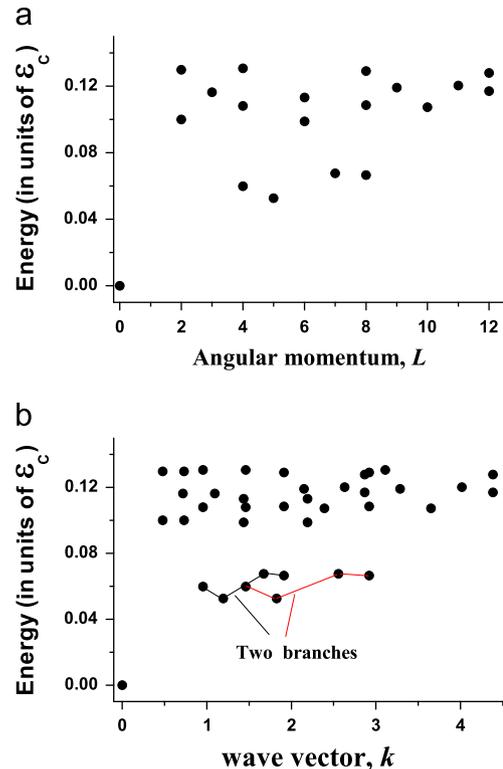


Fig. 3. (Color online) (a) Energy spectrum of the eight-electron $\nu = \frac{1}{3}$ -FQHE system in the $n=1$ LL. The spectrum is evaluated in the spherical geometry with flux quanta $2S=21$, and $\gamma=0.4$. (b) The energy spectrum of the eight-electron $\nu = \frac{1}{3}$ -FQHE system for $n=1$ as a function of the wavevector, k for $\gamma=0.4$. The results are obtained from the energy spectrum in a spherical geometry [panel (a)]. The two branches are shown schematically in (b).

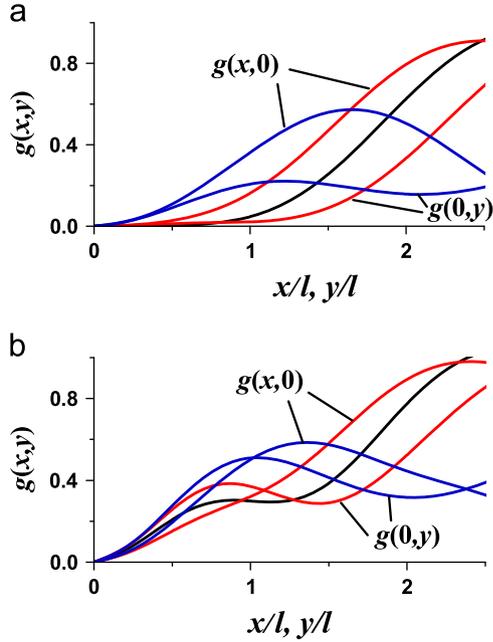


Fig. 4. (Color online) The pair-correlation function $g(x,y)$ for the eight-electron $\nu = \frac{1}{3}$ -FQHE system in the incompressible ground state for different anisotropy parameters: $\gamma = 0$ (black line), $\gamma = 0.2$ (red line), and $\gamma = 0.6$ (blue line). The results are for the spherical geometry with the flux quanta $2S = 21$. We considered two LLs (a) $n=0$ and (b) $n=1$. For an anisotropic system ($\gamma \neq 0$), the correlation functions are shown along the x ($y=0$) and y ($x=0$) directions.

obtained in the spherical geometry and shown in Fig. 3(a). Splitting of the magneto-roton mode was first observed experimentally for conventional semiconductor systems [17] and was suggested [18] to be due to the interaction-induced anisotropy (just as in the present case). Experimental confirmation of this splitting in graphene would be a very important development.

Properties of the incompressible state can also be obtained from the pair-correlation function

$$g(\mathbf{r}) = \int \dots \int d\mathbf{r}_3 \dots d\mathbf{r}_N |\Phi_0(\gamma, 0, \mathbf{r}, \mathbf{r}_3, \dots, \mathbf{r}_N)|^2, \quad (16)$$

where $\Phi_0(\gamma, \mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \dots, \mathbf{r}_N)$ is the N -particle wavefunction of the incompressible ground state. The ground state of the many-particle system is initially calculated in the spherical geometry for a special single-particle basis [16], where the states of such basis are characterized by L_z . Then each basis state of the spherical geometry in the expression for the ground state of the many-electron system is replaced by the corresponding anisotropic state of the planar geometry with the same z component of the angular momentum, $m = L_z$. The resulting pair-correlation function in planar geometry is shown for different values of γ in Fig. 4.

For $\gamma = 0$, the pair-correlation function $g(\mathbf{r})$ is isotropic and depends only on the magnitude of \mathbf{r} . For $n=0$ (Fig. 4a), the incompressible state of the system is described by the Laughlin state with $g(r) \propto r^6$ for small r . For an increase in anisotropy, the correlation function becomes anisotropic and shows quadratic dependence on r , which signifies the suppression of the FQHE gap. Even for $\gamma = 0.2$, when the gap is still large, the pair-correlation function differs strongly from the isotropic case. With increasing anisotropy, $g(r)$ shows local maxima at finite values of r , which suggests a transition to a compressible state with crystalline structure [6]. For $n=1$ (Fig. 4b), due to the presence of both $n=0$ and $n=1$ conventional Landau wavefunctions in a single-electron basis [Eq. (7)], the pair-correlation function has a quadratic dependence on r , even for the isotropic case. For increasing anisotropy, the pair-correlation function, just as for $n=0$, develops additional local maximum for finite

values of r . The anisotropy of the pair-correlation function in $n=1$ is much weaker than that for $n=0$, which suggests a more stable FQHE in the $n=1$ LL, that is consistent with the behavior of the gap in different LLs (Fig. 3). This means that the FQHE state in graphene is less sensitive to anisotropy of the interaction than that in conventional semiconductors.

4. Conclusion

The anisotropy of inter-electron interactions changes the properties of the incompressible FQHE states in graphene. With increasing anisotropy, which is characterized by the parameter γ , the FQHE gap is reduced and at a critical value of $\gamma_c \approx 0.8$ the ground state becomes compressible with zero excitation gap. In isotropic graphene, the FQHEs of comparable strength can be observed at both $n=0$ and $n=1$ Landau levels, where the stronger FQHE, i.e., with the larger FQHE gap, is realized in the $n=1$ Landau level. The anisotropy does not alter the relation between the FQHE gaps in the two Landau levels, $n=0$ and $n=1$, and for a given value of γ the FQHE gap at $n=1$ is the largest. The critical values of the anisotropy parameter $\gamma_c \approx 0.8$ are almost the same for the two Landau levels $n=0$ and $n=1$.

Similar to the conventional systems, the anisotropy in graphene introduces new features in both excitation spectrum and ground state properties of FQHE systems. This is most evident for the low-energy excitation spectrum of the FQHE states in graphene which shows two branches, corresponding to splitting of the magneto-roton excitation mode of the isotropic system. The electron correlation properties of the ground state of the FQHE system are also modified in the presence of the anisotropy. Such changes are visible in the pair-correlation function of the FQHE incompressible state.

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