Theory of Incompressible States in a Narrow Channel

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We report on the properties of a system of interacting electrons in a narrow channel in the quantum Hall effect regime. It is shown that an increase in the strength of the Coulomb interaction causes abrupt changes in the width of the charge-density profile of translationally invariant states. We derive a phase diagram which includes many of the stable odd-denominator states as well as a novel fractional quantum Hall state at the lowest half-filled Landau level. The collective mode evaluated at the half-filled case is strikingly similar to that for an odd-denominator fractional quantum Hall state. [S0031-9007(97)03447-9]

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The existence of incompressible quantum fluid states in a two-dimensional electron system subjected to a strong perpendicular magnetic field has presented us with a profound understanding of the odd-denominator fractional quantum Hall effect (FQHE) [1–4]. Interestingly, such a clear physical understanding of the simplest even-denominator state, viz., the half-filled lowest Landau level, has not yet emerged [3,5–7]. In recent years, study of electron correlations in narrow channels has received increasing attention [8]. Observation of a new incompressible state at a half-filled Landau level in a narrow channel was indeed reported a few years ago [9], and is naturally expected in the Laughlin picture generalized to one dimension [10], where the statistics of the charge carriers should be arbitrary [11]. That observation was remarkable because such a state has, as yet, not been found to appear either in experiments [1,5] or in numerical studies [7] of a two-dimensional electron gas. There have been several attempts to explain the origin of the nonexistence of a stable half-filled quantum Hall state in two dimensions. These include, among other things, a transformation from electrons to fermions with a Chern-Simons field [6]. One other possible explanation was suggested in Refs. [7,12], where it was shown that reduction of the short-range part of the repulsive electron-electron interaction is required to stabilize this state.

Here we report on the results of a model we have developed for the FQHE in a narrow channel (1D-FQHE), where there are a finite number of spin polarized electrons subjected to a strong perpendicular magnetic field, interacting via the long-range Coulomb potential and confined by a potential, which is *parabolic* in one dimension and flat in the other. The electrons are considered to be in a cell whose length in the x direction is denoted by a. The width of the cell depends on the strength of the confining potential relative to the strength of the interactions and also on the length of the cell. Imposing a periodicity condition in the x direction, the system models an infinitely long quantum wire. Our novel results here include abrupt changes in the electron density profile as one moves from one FQH state to another. These stable states, which also include the unique lowest half-filled Landau level, are identified from their gap structures in the excitation spectra.

The total Hamiltonian for the system is

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{\rm int}\,, \tag{1}$$

where $\mathcal{H}_0$ contains the kinetic energy of N electrons of mass m^* and the electrostatic confining potential,

$$\mathcal{H}_0 = \sum_{i=1}^{N}\left[\frac{1}{2m^*}(\mathbf{p}_i - e\mathbf{A}_i)^2 + \frac{1}{2}m^*\omega_0^2 y_i^2\right], \tag{2}$$

and $\mathbf{A}$ is vector potential in *Landau gauge*. The interaction term of the Hamiltonian consists of the Coulomb repulsion of the electrons, the electrostatic energy of the positive background, and the interaction energy between the background and the electrons.

The single-electron wave functions are given by

$$\psi_\kappa(\mathbf{r}) = \left(\frac{1}{a\sqrt{\pi}\,\lambda}\right)^{1/2}\exp\left(ikx - \frac{\hat{y}^2}{2\lambda^2}\right)H_n\left(\frac{\hat{y}}{\lambda}\right), \tag{3}$$

where the magnetic length is defined as $\lambda = (\hbar/m^*\Omega)^{1/2}$, and $\Omega = (\omega_0^2 + \omega_c^2)^{1/2}$, where $\omega_c = eB/m^*$ is the cyclotron frequency, $\kappa = (n, m)$, and

$$\hat{y} = y + \frac{\hbar\omega_c}{m^*\Omega^2}k = y + \frac{2\pi\lambda^2}{\gamma a}m\,,$$

with a dimensionless quantity $\gamma = \sqrt{1 + (\omega_0/\omega_c)^2}$. In (3), H_n is a Hermite polynomial of order n. Along the wire the wave function (3) is just a plane wave with wave vector $k = (2\pi/a)m$. Here m stands for momentum in the direction of the wire. In the lateral y direction the wave function has a Gaussian form. Restricting ourselves in the lowest Landau level, i.e., setting $n = 0$, and ignoring the constant Landau level energy, the single-electron Hamiltonian (2) in the second quantized form is

$$\mathcal{H}_0 = \sum_i \frac{\hbar^2 k_i^2}{2m^*} \frac{\omega_0^2}{\Omega^2} a_i^\dagger a_i = \sum_i \mathcal{E}_i a_i^\dagger a_i ,$$

where $a_i^\dagger$ (a_i) is the creation (annihilation) operator of a state i.

In the noninteracting ground state, N electrons occupy the lowest N available single-particle levels. It is reasonable to require that the electron density in that state be symmetric around the $y = 0$ axis, i.e., the total momentum $M = \sum_j m_j = 0$. This symmetry condition holds for *odd* number of electrons if m is an *integer*, and for *even* number of electrons if m is a *half-odd integer*. Thus, for an odd number of electrons, we have *periodic* boundary conditions along the wire and *antiperiodic* boundary conditions for an even number of electrons. When interelectron interactions are introduced in the system the electrons start to avoid each other. As interactions increase with respect to the kinetic energy, electrons begin to also occupy higher levels in order to reduce their mutual repulsion. Consequently, states other than $M = 0$ are also realized as ground states. However, if the ground state has $M \neq 0$, the system is not expected to be in a fractional quantum Hall state [4].

The Coulomb matrix elements in the present model are obtained from

$$\begin{aligned}\mathcal{A}_{m_1,m_2,m_3,m_4} &= \frac{1}{2} \int d\mathbf{r}_1 d\mathbf{r}_2 \psi_{m_1}^*(\mathbf{r}_1)\psi_{m_2}^*(\mathbf{r}_2) v(\mathbf{r}) \psi_{m_3}(\mathbf{r}_2)\psi_{m_4}(\mathbf{r}_1) \\ &= \frac{1}{2} \frac{e^2}{\epsilon\lambda} \exp\left[-\frac{1}{2}\left(\frac{2\pi}{\gamma a}\right)^2 (m_1 - m_4)^2\right] \int dq_y' \frac{\exp[i2\pi(m_3 - m_1)q_y'] \exp[-\frac{1}{2}(\gamma a q_y')^2]}{\sqrt{[\frac{2\pi(m_1-m_4)}{\gamma a}]^2 + (a q_y')^2}} \\ &\quad \times \delta_{m_1+m_2,m_3+m_4} ,\end{aligned} \tag{4}$$

where the length is measured in units of λ, $\mathcal{E}_c = e^2/\epsilon\lambda$ gives a measure of the interaction energy, and the dimensionless integration variable is $q_y' = q_y \lambda^2/(\gamma a)$. When $m_1 = m_4$, the integral in the second term of (4) does not converge due to the long-range nature of the Coulomb potential. To cancel out this divergence we have two choices: We can either use a truncated Coulomb potential [13] or neutralize the system by embedding the wire into a positively charged background. We prefer the latter procedure because then the long-range effects of the Coulomb force are included in our calculations.

Let us first examine how the translationally invariant state, i.e., the $M = 0$ state, changes when we change the strength of the interactions with respect to kinetic and potential energies of the electrons, i.e., $\mathcal{E}_c/E_0$ [where $E_0 = (\hbar^2/2m^*\lambda^2)(\omega_0^2/\Omega^2)$ is the energy unit] and the length of the cell a. As we vary $\mathcal{E}_c/E_0$ while keeping a fixed, the expectation values of the kinetic and potential energies change abruptly from one value to another. As the calculation is repeated for other fixed values of a we obtain Figs. 1(a) and 1(b) for $\langle\mathcal{H}_0\rangle$ and $\langle\mathcal{H}_{\rm int}\rangle$, respectively. The expectation values show rich structures in the parameter space spanned by $a = 5, \ldots, 12.4$ and $\mathcal{E}_c/E_0 = 0, \ldots, 80$. The two energies $\langle\mathcal{H}_0\rangle$ and $\langle\mathcal{H}_{\rm int}\rangle$ jump in opposite directions, and therefore the net change in total energy does not clearly show the sudden changes in the $M = 0$ state. However, for a much longer system (at a fixed linear density), we expect sharper first-order transitions between the different phases.

As the jump occurs in the parameter space spanned by $\mathcal{E}_c/E_0$ and a, it indicates a change in the $M = 0$ state. One earlier work identified the filling factors ($\nu = N/N_s$ where N_s is the Landau level degeneracy) $\nu = \frac{2}{3}$, $\nu = \frac{1}{3}$, and $\nu = \frac{1}{5}$ FQHE states in a system of six electrons interacting via a truncated Coulomb potential and calculating the overlap with the Laughlin-like wave functions [13]. These states are also realized in our system with real long-range Coulomb potential. The state at $\nu = \frac{1}{3}$ can also be characterized by calculating the overlap between the Coulomb $\frac{1}{3}$ state and Haldane's pseudopotential $\frac{1}{3}$ state [14]. We have checked this overlap in our present system and found it to vary between the values 0.83 and 0.89 at $a = 9.5$.

In our quest for a stable half-filled Landau level, we are particularly interested to know what happens in between the well-established FQHE states. For example, what are the states realized in between the FQHE states $\nu = \frac{2}{3}$ and $\nu = \frac{1}{3}$? In this region there are clear jumps in both $\langle\mathcal{H}_0\rangle$ and $\langle\mathcal{H}_{\rm int}\rangle$. To get further insight into the $M = 0$ states realized in the wire, we have investigated the problem of how the electron density is modified when we change $\mathcal{E}_c/E_0$ for a fixed value of a. In the x direction the charge density is constant while in the lateral y direction it is modified because of the finite width of the system. Electron density at $\mathbf{r}$ is evaluated numerically from

$$\rho(\mathbf{r}) = \sum_{i,j=1}^{\infty} \psi_i^*(\mathbf{r})\psi_j(\mathbf{r}) a_i^\dagger a_j .$$

Let us choose a particular value of a, say $a = 8$, and see how the density profile of the translationally invariant state changes as a function of $\mathcal{E}_c/E_0$. In Fig. 2(a), we show the charge densities as a function of $\mathcal{E}_c/E_0$. With increasing $\mathcal{E}_c/E_0$, the width of the charge-density profile changes abruptly from one value to the other. Calculating the effective filling factor from $\nu = 2\pi\lambda^2 n$ (where n is the number of electrons per unit area), and taking the width as full width at half-maximum, we get the filling factors 0.98, 0.71, 0.56, …, 0.51 and 0.42 which are very close to the values $\nu = 1, \frac{2}{3}, \frac{1}{2}$ and $\nu = \frac{2}{5}$. Repeating the

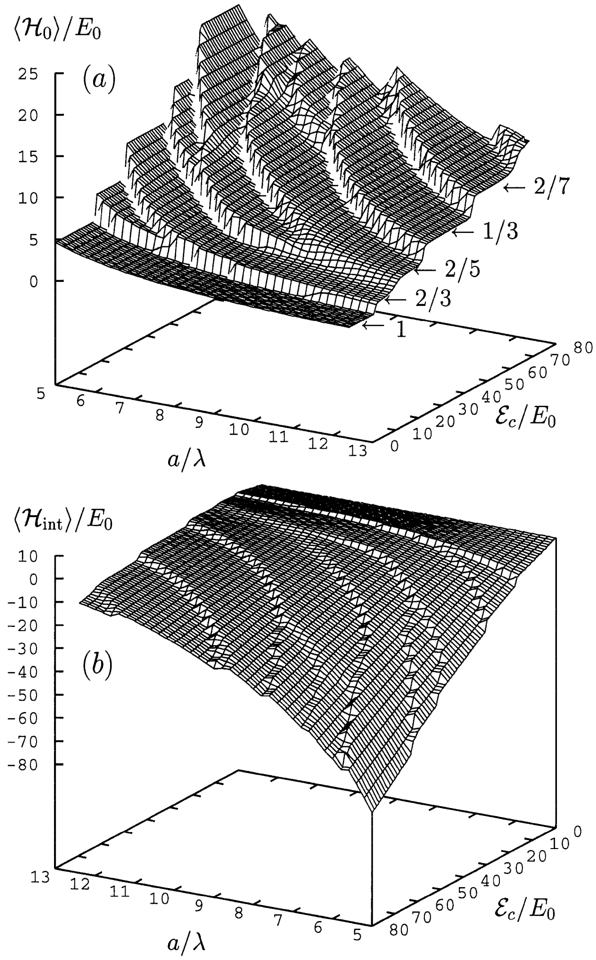

FIG. 1. Expectation value of (a) kinetic energy per particle and (b) interaction energy per particle, as a function of $\mathcal{E}_c/E_0$ and length of the cell for the state $M = 0$. The effective filling factors $\nu = \frac{1}{3}, \frac{2}{3}, \frac{2}{5}, \frac{2}{7}$, and $\nu = 1$ are also indicated.

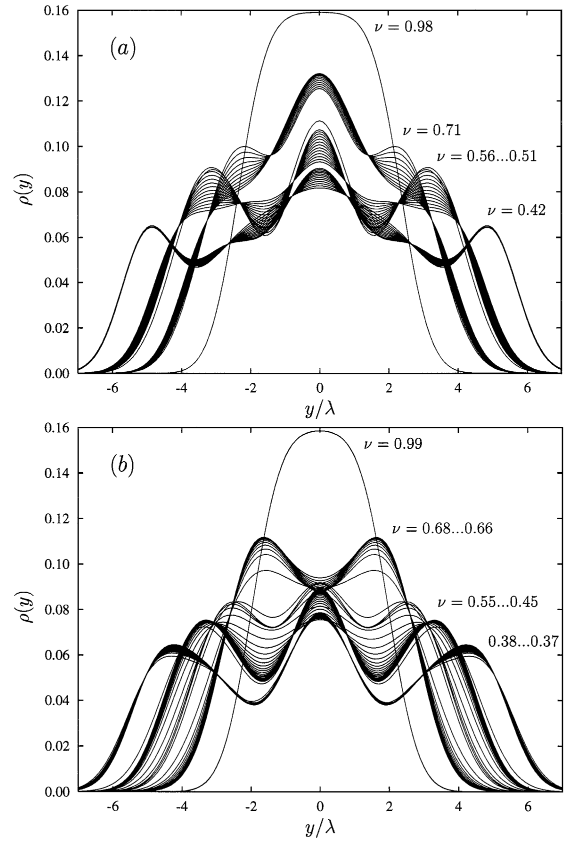

FIG. 2. (a) Electronic densities in the lateral direction at $a = 8$ and for $M = 0$ states. (b) Similar results for $a = 9.5$. The effective filling factors are shown in the figure.

same calculation at $a = 9.5$, we get the densities shown in Fig. 2(b). The effective filling factors for this value of a are 0.99 and $0.68, \ldots, 0.66$, which suggest that these states are $\nu = 1$ and $\nu = \frac{2}{3}$, respectively. The state which has the effective filling factor $0.38, \ldots, 0.37$ is identified as a $\nu = \frac{1}{3}$ state by the overlap calculation.

In Fig. 3 we show a phase diagram for the 1D-FQHE states. The diagram is obtained by systematically seeking those points in the parameter space spanned by a and $\mathcal{E}_c/E_0$, where the ground state has *zero total momentum*. We then plot the energy gap between this ground state and the first excited state. In Fig. 3 the area of a filled dot is proportional to that gap. The phase diagram consists of separate regions of several FQHE states. Remarkably, there is a distinct region for the *even-denominator* state $\nu = \frac{1}{2}$. The area of this region is, of course, much smaller than those with odd-denominator states. But, given the total absence of the $\frac{1}{2}$ state in a single-layer system, this observation is rather unique. Figure 4 depicts the energy spectra for the states $\nu = \frac{2}{3}, \frac{1}{2}, \frac{2}{5}$ and $\nu = \frac{1}{3}$. These states are chosen from the phase diagram (Fig. 3) at the points where the gap appears to be the largest. The ubiquitous incompressible gaps in the spectra makes the analogy with those in the corresponding two-dimensional systems quite obvious. The novel result here again is, of course, the signature of incompressibility in the energy spectrum for the lowest half-filled Landau

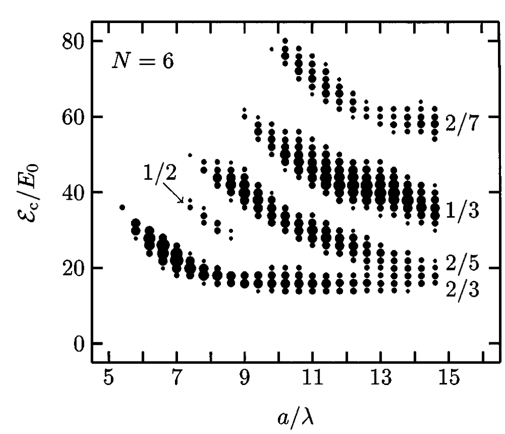

FIG. 3. Phase diagram for the FQHE states at the effective filling factors $\nu = \frac{1}{3}, \frac{2}{3}, \frac{2}{5}, \frac{1}{2}, \frac{2}{7}$ indicated in the figure.

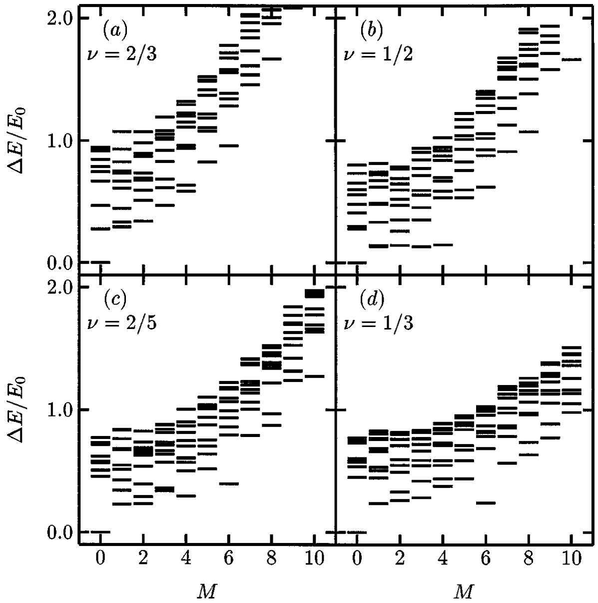

FIG. 4. Energy spectra calculated at $(a, \mathcal{E}_c/E_0)$ and ν: (a) (6.8, 24), $\nu = \frac{2}{3}$, (b) (7.6, 36), $\nu = \frac{1}{2}$, (c) (9.2, 40), $\nu = \frac{2}{5}$, and (d) (12.2, 42), $\nu = \frac{1}{3}$.

level, which is a hallmark of the fractional quantum Hall state at that filling factor. The size of the gap at $\nu = \frac{1}{2}$ is comparable to that at $\frac{2}{3}$ and should be observable in experiments. There are also experimental indications [9] that the gap size at $\nu = \frac{1}{2}$ generally depends on the width of the constriction. These issues and the problem of how the gap vanishes when we approach two dimensionality will be the subject of future publications.

Between the stable 1D-FQHE regions in the phase diagram, the system is in states with $M \neq 0$. This suggests that the symmetry changes between ground states in different regions of the phase diagram. One possible broken symmetry state might be a phase in which the density is no longer symmetric around $y = 0$, i.e., some of the density is displaced from the left side of the channel to the right side (or vice versa) spontaneously [15]. Interestingly, for values of a where broken symmetry states appear, the minimum value of the excitation gap seems to collapse towards zero [16]. However, there can never be a true long-range order in the spacing of electrons along the wire (in the thermodynamic limit). In fact, in a one-dimensional system, no sharp distinction exists between a Wigner crystal with quantum fluctuations and a spinless Luttinger liquid. In either case, the density-correlation function has a power-law singularity at a wave vector equal to twice the Fermi wave vector, and the exponent of the singularity varies continuously with the interaction parameters. These issues will be discussed elsewhere [16].

In conclusion, we have investigated the properties of a system of electrons interacting via the long-range Coulomb interaction in a narrow channel and in the quantum Hall regime. As the interaction strength is increased, abrupt jumps occur in the expectation values of the kinetic and potential energies of translationally invariant states. The width of the charge-density profile also shows similar abrupt changes. We present the phase diagram of the stable 1D-FQHE states. In addition to various odd-denominator filling factors which are well established in the two-dimensional systems, we find that, in a region of the parameter space, the lowest half-filled Landau level also appears as a stable incompressible state. We also present the energy spectra of those incompressible states. The low-lying collective modes at $\nu = \frac{1}{2}$ are strikingly similar to those of an odd-denominator FQHE state.

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