reprinted from

Volume 1

Basic Properties of Semiconductors

Volume editor

P. T. LANDSBERG

Faculty of Mathematical Studies
University of Southampton
Southampton SO9 5NH, UK

1992

NORTH-HOLLAND
AMSTERDAM · NEW YORK · LONDON · TOKYO

CHAPTER 17

The Quantum Hall Effect

TAPASH CHAKRABORTY

Institute for Microstructural Sciences
National Research Council
Montreal Road, M-50
Ottawa, Canada K1A 0R6

Handbook on Semiconductors
Completely Revised Edition
Edited by T.S. Moss
Volume 1, edited by P.T. Landsberg

Contents

1. Introduction

The quantization of the Hall effect discovered by von Klitzing et al. (1980) is a remarkable macroscopic quantum phenomenon which occurs in two-dimensional electron systems at low temperatures and strong perpendicular magnetic fields. Under these conditions, the Hall conductivity exhibits plateaus at integral multiples of e^2/h (a universal constant). The striking result is the accuracy of the quantization (better than one part in ten million) which is totally indifferent to impurities or geometric details of the two-dimensional system. Each plateau is accompanied by a deep minimum in the diagonal resistivity, indicating a dissipationless flow of current. In 1982, there was yet another surprise in this field. Working with much higher mobility samples, Tsui et al. (1982) discovered the fractional quantization of the Hall conductivity. The physical mechanism responsible for the integer quantum Hall effect (IQHE) and the fractional quantum Hall effect (FQHE) are quite different, despite the apparent similarity of the experimental results. In the former case, the role of the random impurity potential is quite decisive, while in the latter case electron–electron interaction plays a predominant role resulting in a unique collective phenomenon.

In the following sections, we shall briefly describe the theoretical and experimental developments in the QHE. It should be mentioned, however, that the QHE has been one of the most active fields of research in condensed matter physics for over a decade. It is, therefore, quite impossible to describe here all the details of the major developments. Our aim in this chapter is to touch upon the most significant theoretical and experimental work to construct a reasonably consistent picture of the QHE. For more details on the topics discussed, the reader is encouraged to read the original work cited here and some of the reviews available in the literature (Aoki 1987, Chakraborty and Pietiläinen 1988a, Hajdu and Landwehr 1985, Halperin 1983, MacDonald 1989, Morandi 1988, Prange and Girvin 1987, Rashba and Timofeev 1986).

2. Two-dimensional electron gas

The major impetus in the studies of the QHE is due to experimental realization of almost ideal two-dimensional electron systems. The electrons are dynamically two-dimensional because they are free to move in only two spatial dimensions. In the third dimension, they have quantized energy levels [in reality, the wave functions have a finite spatial extent in the third dimension (Ando et al. 1982)]. In the following, we provide a very brief discussion on the systems where the electron layers are created. For details see the reviews by Ando et al. (1982) and Störmer (1984).

2.1. *Systems with a two-dimensional electron gas*

Electron layers have been created in many different systems. Electrons on the surface of liquid helium provides an almost ideal two-dimensional system (Grimes 1978, Monarkha and Shikin 1982). They are trapped on the surface by a combination of an external field and an image potential. The electron concentration in this system is, however, very low (10^5–10^9 cm^{-2}) and the system behaves classically. The high-density electron systems where the QHE is usually observed are typically created in the metal-oxide–semiconductor field effect transistor (MOSFET) and in semiconductor heterojunctions.

2.1.1. *Inversion layers in Si-MOSFET structures*

A schematic picture of an n-channel Si-MOSFET is shown in fig. 1a. The system consists of a semiconductor (p-Si) which has a plane interface with a thin film of insulator (SiO_2), the opposite side of which carries a metal gate electrode. Application of a voltage (gate voltage V_G) between the gate and the Si/SiO_2 interface results in bending of the electron energy bands. For a strong enough electric field, as the bottom of the conduction band is pushed down below the Fermi energy E_F, electrons accumulate in a two-dimensional quasi-triangular potential well close to the interface (fig. 1b). As the width of the well is small (~ 50 Å), electron motion perpendicular to the interface is quantized but the electrons move freely parallel to the interface. In the plane, the energy spectrum is

$$\varepsilon_i(\boldsymbol{k}) = \varepsilon_i^0 + \frac{\hbar^2 k_{\|}^2}{2m^*},$$

where m^* is the effective mass of the electrons, $k_{\|}$ is the two-dimensional wave vector and ε_i^0 is the bottom of the corresponding subband. The system is called an inversion

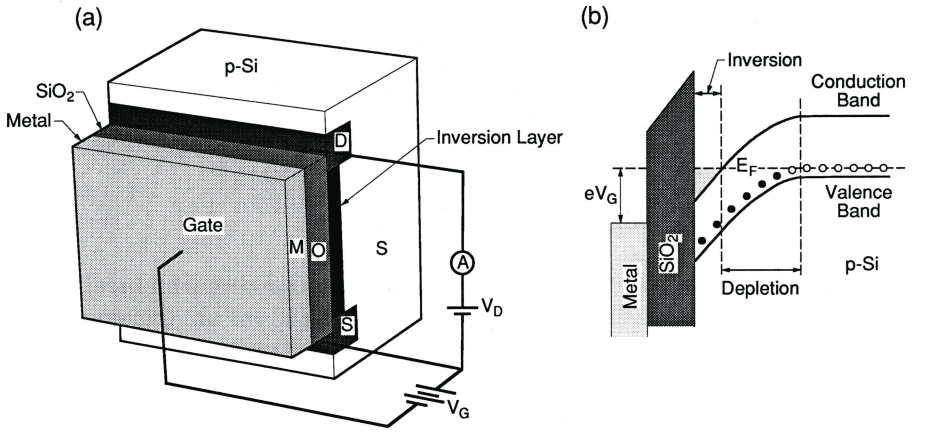

Fig. 1. (a) Schematic view of an Si-MOSFET and (b) energy level diagram.

layer because here the charge carriers are the electrons while the semiconductor is p-type.

At low temperatures ($kT \ll \Delta E$, the energy spacing) the electrons are trapped in the lowest subband and the system is purely two-dimensional. The MOSFET is quite useful in the present study because by varying the gate voltage the electron concentration can be varied within a wide range ($n_0 \sim 0$–10^{13} cm^{-2}).

2.1.2. Electron layers in semiconductor heterostructures

Two-dimensional electron layers are also created in semiconductor heterostructures at a nearly perfectly lattice-matched semiconductor/semiconductor interface. One such widely used system is the GaAs/$Al_xGa_{1-x}As$ ($0 < x \leqslant 1$) heterostructure. The lattice constants of GaAs and $Al_xGa_{1-x}As$ are almost the same so that the interface is nearly free from any disorder. The band gap of the alloy is wider than that of GaAs and it increases with the aluminum concentration x. Carriers in the neighborhood of the heterojunction transfer from the doped AlGaAs alloy across the interface to the low-lying band edge states of the narrow band gap material (GaAs). The electric field due to the charge transfer bends the energy bands as shown in fig. 2. A quasi-triangular potential well (~100 Å) formed in the GaAs traps the electrons as two-dimensional carriers.

The mobile carriers are spatially separated from their parent ionized impurities via modulation doping. This leads to very high carrier mobilities and, in fact, the FQHE was discovered in these high-mobility GaAs-heterostructures (Störmer 1984). However, unlike MOSFETs, the electron concentration in heterostructures can be varied only within a very narrow range. Carrier densities in these systems typically range from 1×10^{11} cm^{-2} to 1×10^{12} cm^{-2}.

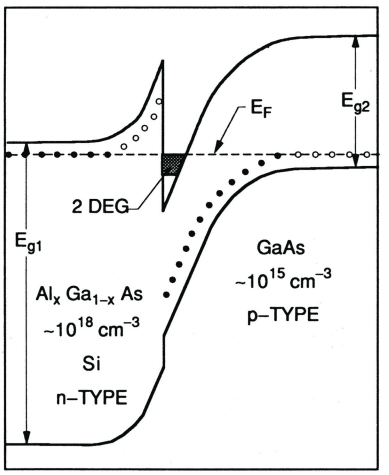

Fig. 2. Energy diagram at a GaAs-heterostructure interface.

2.2. *Electrons in a strong magnetic field*

Let us begin with the problem of a free electron (with effective mass m^*) in a uniform magnetic field B. The Hamiltonian is then written

$$\mathcal{H}_0 = (\Pi_x^2 + \Pi_y^2)/2m^*, \tag{1}$$

where $\boldsymbol{\Pi} = -i\hbar\nabla + (e/c)\boldsymbol{A}$ is the kinetic momentum and the vector potential $\boldsymbol{A}$ is related to the magnetic field as $\boldsymbol{B} = \nabla \times \boldsymbol{A}$. Following Kubo et al. (1965), we introduce the center coordinates of the cyclotron motion (X, Y) as

$$X = x - \xi, \qquad Y = y - \eta, \tag{2}$$

where

$$\xi = (c/eB)\Pi_y, \qquad \eta = -(c/eB)\Pi_x, \tag{3}$$

are the relative coordinates. It can be easily seen that (ξ, η) represents a cyclotron motion with frequency

$$\omega_c = \frac{eB}{m^*c} \quad \text{(cyclotron frequency)}. \tag{4}$$

Defining the magnetic length

$$l_0 \equiv \left(\frac{\hbar c}{eB}\right)^{1/2} \quad \text{(cyclotron radius)} \tag{5}$$

and from the commutation relation

$$[\xi, \eta] = -il_0^2,$$

it is clear that ξ and η are subject to an uncertainty of order l_0. The Hamiltonian (1) is now rewritten in terms of (ξ, η) as

$$\mathcal{H}_0 = \frac{\hbar\omega_c}{2l_0^2}(\xi^2 + \eta^2), \tag{6}$$

whose eigenenergies are the discrete *Landau levels* (Fock 1928, Landau 1930)

$$E_n = (n + \tfrac{1}{2})\hbar\omega_c, \quad n = 0, 1, 2, \ldots . \tag{7}$$

The Hamiltonian (6) does not contain (X, Y), which means that electrons in cyclotron motion with different center coordinates have the same energy. The center coordinates also follow the commutation rule, $[X, Y] = il_0^2$.

Choosing now the Landau gauge such that the vector potential $\boldsymbol{A}$ has only one nonvanishing component, say, $A_y = Bx$, the Hamiltonian is

$$\mathscr{H}_0 = \frac{1}{2m^*}\left[p_x^2 + \left(p_y + \frac{eB}{c}x\right)^2\right]. \tag{8}$$

The variables are easily separable and an eigenfunction is written in the form

$$\varphi = e^{ik_y y}\,\chi(x), \tag{9}$$

where the usual identification is made, $p_y = -i\hbar\,\partial/\partial y \rightarrow \hbar k_y$. The function $\chi(x)$ is the eigenfunction of the time-independent Schrödinger equation

$$-\frac{\hbar^2}{2m^*}\chi'' + \tfrac{1}{2}m^*\omega_c^2(x - X)^2\chi = E\chi(x), \tag{10}$$

where $X = -k_y l_0^2$. The above equation is easily recognized as the Schrödinger equation corresponding to a *harmonic oscillator* of spring constant $\hbar\omega_c = \hbar^2/m^* l_0^2$, with equilibrium point at X.

The eigenfunction (ignoring the normalization factor) is now written

$$\varphi_{nX} = e^{ik_y y}\exp[-(x - X)^2/2l_0^2]\,H_n[(x - X)/l_0] \tag{11}$$

with H_n the Hermite polynomial. The functions are extended in y and localized in x. The localization remains unaffected under a gauge transformation. When the system is confined in a rectangular cell with sides L_x and L_y, the degeneracy of each Landau level (number of allowed states) is, in fact, the number of allowed values of k_y, such that the center X lies between 0 and L_x. With use of periodic boundary conditions we get, $k_y = 2\pi n_y/L_y$, with n_y an integer. The allowed values of n_y are then determined by the condition

$$X = \frac{2\pi n_y}{L_y}l_0^2, \quad 0 < X < L_x. \tag{12}$$

The degeneracy N_s can then be expressed in terms of the magnetic length l_0 as

$$N_s = \frac{L_x L_y}{2\pi l_0^2}. \tag{13}$$

Equation (13) can also be reexpressed in terms of the magnetic flux $\boldsymbol{\Phi}$ and the flux quantum $\boldsymbol{\Phi}_0 = hc/e$ as

$$N_s = \frac{e}{hc}\boldsymbol{\Phi} = \frac{\boldsymbol{\Phi}}{\boldsymbol{\Phi}_0}. \tag{14}$$

The Landau-level degeneracy is thus the total number of flux quanta in the external magnetic field. One other important quantity is the dimensionless density of the electrons expressed as the *filling factor* of the Landau level

$$\nu = 2\pi l_0^2 n_0, \tag{15}$$

where n_0 is the electron density in the system.

Thus far, we have ignored the presence of any impurities in the system. In a more general case, the Hamiltonian is written as

$$\mathscr{H} = \mathscr{H}_0 + U(\boldsymbol{r}), \tag{16}$$

where $U(\boldsymbol{r})$ is the electron–impurity interaction. Following Kubo et al. (1965), the equation of motion for (X, Y) can be derived as

$$\begin{aligned} \dot{X} &= \frac{\mathrm{i}}{\hbar}[\mathscr{H}, X] = \frac{\mathrm{i}}{\hbar}[U, X] = \frac{l_0^2}{\hbar}\frac{\partial U}{\partial y}, \\ \dot{Y} &= \frac{\mathrm{i}}{\hbar}[\mathscr{H}, Y] = \frac{\mathrm{i}}{\hbar}[U, Y] = -\frac{l_0^2}{\hbar}\frac{\partial U}{\partial x}, \end{aligned} \tag{17}$$

where we have employed the commutation relation: $[x, \eta] = -[y, \xi] = \mathrm{i}l_0^2$. Due to the presence of the impurity potential, the degeneracy of the states with different (X, Y) is lifted and the Landau levels instead of being a series of δ-functions broaden into bands (fig. 3).

The density of states $D(E)$ of the two-dimensional electron system is an important quantity for the understanding of the QHE. Several authors have contributed theoretically and experimentally to our present understanding of the density of states (DOS) in the presence of a random potential (Ioffe and Larkin 1981, Wegner 1983, Brezin et al. 1984, Ando and Uemura 1974, Ando 1983a, 1984, Ando and Aoki 1985, Aoki and Ando 1985). The DOS can be determined by measuring thermodynamic quantities like magnetization (Eisenstein et al. 1985), electron heat capacity (Gornik et al. 1985) and quantum oscillations of the chemical potential (Pudalov et al. 1985), from magnetocapacitance measurements (Smith et al. 1986), from activated conduction (Stahl et al. 1985), from the temperature dependence of the slope of the Hall plateau (Wei et al. 1985a) or from an analysis of the shape of the Hall plateau (Pudalov and Semenchinsky 1985). A review of the experimental results is available in the literature (Gornik 1987). The results of all these studies can be stated as follows: the DOS between the $D(E)$ peaks is approximately constant and is a significant fraction of the value at zero magnetic field. The width of each peak is $\Gamma \sim B^{1/2}$. It is also established now that there are localized states in the tails of $D(E)$ and extended

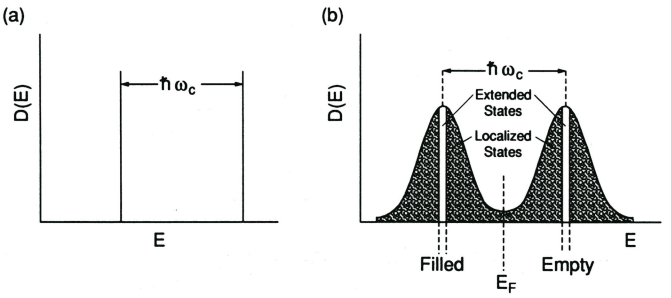

Fig. 3. Density of states versus energy. (a) impurity-free system, $U = 0$. (b) $U \neq 0$, with mobility edges and the localized regions.

states which carry the Hall current located in the region of the maximum of $D(E)$. The regions of localized states are known as the *mobility gaps* and their boundaries with the regions of extended states are called the *mobility edges.* For further details, see § 3.3.1.

3. Integral quantum Hall effect

Before the discovery of the IQHE by von Klitzing et al. (1980), there were experimental indications of the existence of such an effect. Plateau-like behavior was actually observed in ρ_{xy} (Englert and von Klitzing 1978) and in σ_{xy} (Wakabayashi and Kawaji 1980). Accurate quantization of the Hall plateau was not achieved in those experiments. The possibility of ρ_{xy} quantization was also considered theoretically by Ando et al. (1975).

3.1. Experimental work

In fig. 4, we present the experimental results of von Klitzing et al. (1980) and von Klitzing (1982, 1986) for a Si-MOSFET inversion layer in a magnetic field of $B = 19$ T. The diagonal resistance R_{xx} ($\approx \rho_{xx}$) and the Hall resistance R_{xy} ($\approx R_H$) are plotted as a function of gate voltage V_G ($\propto$ electron concentration). The diagonal

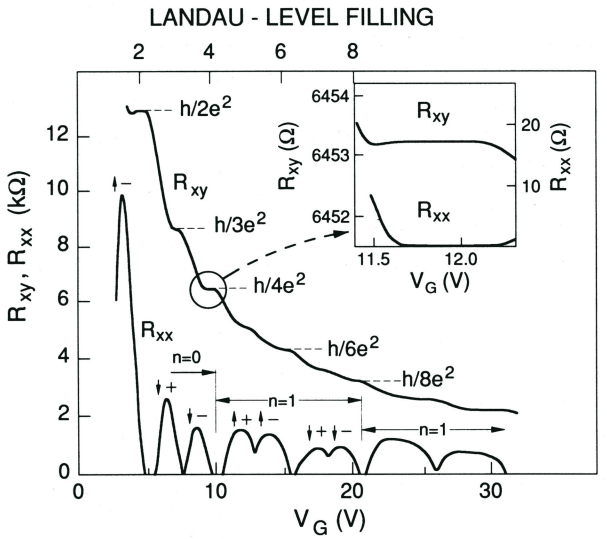

Fig. 4. The quantum Hall effect observed in Si(100) MOS inversion layer in a magnetic field of $B = 19$ T at temperature $T = 1.5$ K. The diagonal resistance and the Hall resistance are shown as a function of gate voltage ($\propto$ electron concentration). The oscillations in R_{xx} are labeled by the Landau-level index (n), spin ($\uparrow, \downarrow$) and the valley ($+, -$). The upper scale indicates the Landau-level filling described in the text, and the inset shows the details of a plateau for $B = 13.5$ T.

resistance is seen to vanish at different regions of V_G, indicating a current flow without any dissipation. In the same regions, R_{xy} develops *plateaus* with $R_{xy} = h/ne^2$ (n integer). The quantization condition on the plateaus are found to be obeyed with *extreme accuracy*. The experimental accuracy so far achieved is better than one part in 10^7 while resistivities as low as $\rho_{xx} < 10^{-10}\Omega/\square$ have been established (Störmer 1984). Another interesting feature of the above findings is that the quantization condition of the conductivity is very insensitive to the details of the sample (geometry, amount of disorder, etc.). The quantized Hall resistance is more stable and more reproducible than any wire resistor and since 1 January 1990, the quantized Hall resistance has been used as an international reference resistor. The Hall resistance of a quantized plateau is expressed as $R_H = R_K/n$ and $R_K \approx 25\,813\,\Omega$ is the *von Klitzing constant* which appears to be a universal quantity (Taylor 1989, Quinn 1989).

The accurate quantization of the Hall resistance can be used to determine the value of the fine-structure constant α (von Klitzing et al. 1980). This is a quantity of fundamental importance in quantum electrodynamics and is related to the Hall resistance in the manner

$$\alpha = \tfrac{1}{2}\mu_0 c \frac{e^2}{h} = \tfrac{1}{2}\mu_0 c (R_K)^{-1},$$

where μ_0 is the permeability of the vacuum and c is the velocity of light.

After the initial experiment by von Klitzing et al., there were several other experiments demonstrating integral quantization in a variety of systems. Tsui and Gossard (1981) observed the effect in a GaAs-heterostructure. This system was also used by other authors (Paalanen et al. 1982, Ebert et al. 1982, von Klitzing 1986). Because of the small effective mass m^* of the electrons in GaAs $[m^*_{Si}/m^*_{GaAs} > 3]$, the Landau-level splitting is larger compared to that in Si and the high-quality of the interface results in a high mobility of the two-dimensional electrons (fig. 5). In addition, the IQHE has been observed in systems like InGaAs/InP (Nicholas et al. 1982, Guldner et al. 1982, 1986), HgTe/CdTe (Kirk et al. 1986) and two-dimensional electrons and holes formed in InAs/GaSb heterostructures (Mendez et al. 1985), in GaAs heterostructures (Störmer et al. 1983a) and in Si-MOSFETs (Gusev et al. 1984).

It is interesting to note that the quantum Hall effect presents a very special situation viz. the conductivity σ_{xx}, describing the current density along the electric field, and resistivity ρ_{xx}, defining the electric field strength along the current path, vanish simultaneously. In a two-dimensional system, in the presence of a magnetic field, the current density $\boldsymbol{j}$ is related to an electric field $\boldsymbol{E}$ by

$$\boldsymbol{j} = \hat{\sigma}\boldsymbol{E}, \quad \boldsymbol{E} = \hat{\rho}\boldsymbol{j}, \tag{18}$$

where $\hat{\sigma}$ is the conductivity tensor and $\hat{\rho} = (\hat{\sigma})^{-1}$ is the resistivity tensor. They are defined as

$$\hat{\rho} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ -\sigma_{xy} & \sigma_{xx} \end{pmatrix}^{-1} = \frac{1}{\sigma_{xx}^2 + \sigma_{xy}^2} \begin{pmatrix} \sigma_{xx} & -\sigma_{xy} \\ \sigma_{xy} & \sigma_{xx} \end{pmatrix}, \tag{19}$$

where we have used the Onsager reciprocity relations (Kubo 1957), $\sigma_{xx} = \sigma_{yy}$ and $\sigma_{yx} = -\sigma_{xy}$. Similarly,

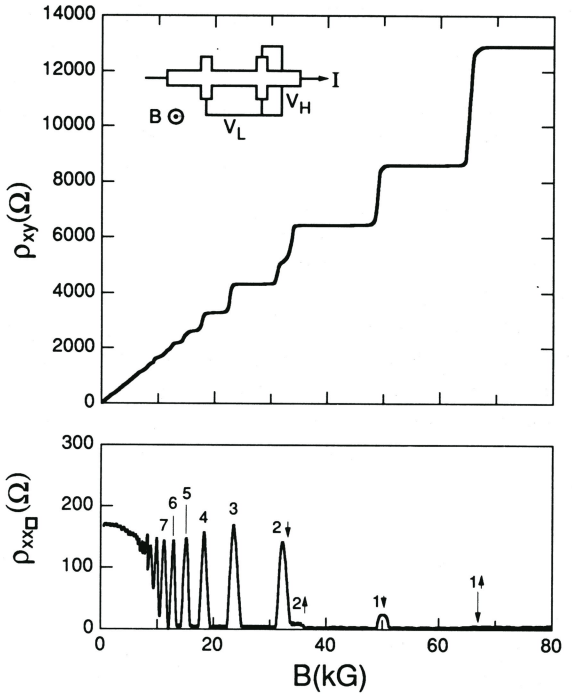

Fig. 5. The Hall resistivity ($\rho_{xy} = V_H/I$) and magnetoresistivity ($\rho_{xx} = V_L/I$) of a modulation-doped GaAs-heterostructure. The inset shows the sample configuration.

$$\hat{\sigma} = \frac{1}{\rho_{xx}^2 + \rho_{xy}^2}\begin{pmatrix} \rho_{xx} & -\rho_{xy} \\ \rho_{xy} & \rho_{xx} \end{pmatrix}. \tag{20}$$

Experimental results show that when a plateau appears, $\rho_{xx} = 0$, which means that, $\sigma_{xx} = \rho_{xx}/(\rho_{xx}^2 + \rho_{xy}^2) = 0$, provided that $\rho_{xy} \neq 0$, and $\sigma_{xy} = -\rho_{xy}/(\rho_{xx}^2 + \rho_{xy}^2) = -1/\rho_{xy}$. For a sample of length L in each dimension, the Hall resistance R_H is related to the Hall resistivity ρ_{xy} as $R_H = -\rho_{xy} L^{2-\delta}$, where δ is the dimensionality. In two dimensions, resistance and resistivity are, therefore, the same quantity. The dimension of R_H is usually expressed as $\Omega/\square$.

3.1.1. Classical Hall effect

The motion of an electron moving classically in crossed electric (E_x) and magnetic (B_z) fields is a superposition of a circular motion with frequency ω_c and a uniform drift perpendicular to E_x with a velocity $\boldsymbol{v} = (0, v_D)$, $v_D = cE_x/B$ (Störmer and Tsui 1983). The resulting orbit is a *trochoid* and is a consequence of the Lorentz force. The current density is then $\boldsymbol{j} = en_0\boldsymbol{v}$. From the discussion above, we can readily deduce that, $\sigma_{xx} = 0$ and $\sigma_{xy} = -n_0 ec/B$. The classical Hall conductivity as a function of electron concentration does not show any quantization. In order to introduce scattering due to random potentials in the system, one adds a term $m^*\boldsymbol{v}/\tau$, where τ

is the scattering relaxation time, to the equation of motion. The resulting diagonal and Hall conductivities are

$$\sigma_{xx} = \frac{n_0 e^2 \tau}{m^*} \frac{1}{1+(\omega_c \tau)^2}; \qquad \sigma_{xy} = -\frac{n_0 ec}{B} \frac{(\omega_c \tau)^2}{1+(\omega_c \tau)^2} = -\frac{n_0 ec}{B} + \frac{\sigma_{xx}}{\omega_c \tau}. \tag{21}$$

The free-electron result is trivially recovered by considering the limit $\omega_c \tau \to \infty$. As we shall see below, this scattering-free situation will occur only in very special circumstances.

3.1.2. Quantum mechanical approach

The quantum mechanical results for the dynamics of an electron in a magnetic field are presented in § 2.2. In the case of an electron in crossed electric and magnetic fields, the eigenstates (7) and (9) can be written as (Brown 1968)

$$\begin{aligned} E_{nX} &= (n+\tfrac{1}{2})\hbar\omega_c + eE_x X - \tfrac{1}{2}m^* v_D^2, \\ \varphi_{nX} &= e^{ik_y y}\, \chi(x - X + v_D/\omega_c). \end{aligned} \tag{22}$$

These states carry a current in the y-direction. The contribution to the current from an occupied state is

$$j_y = -e\left\langle nX \left| \frac{1}{m^*}\left(p_y + \frac{eBx}{c}\right) \right| nX \right\rangle = \frac{c}{B}\frac{\partial E_{nX}}{\partial X} = ev_D \tag{23}$$

since, $\dot{\boldsymbol{r}} = (1/m^*)[\boldsymbol{p} - (e/c)\boldsymbol{A}]$. All the electronic states $|nX\rangle$ carry the same Hall current. The total current is then $j_y = n_0 e v_D$ and the Hall conductivity, as discussed above, is readily obtained as $\sigma_{xy} = -n_0 ec/B$. From the definition of the filling factor (15), we get the quantization as: $\sigma_{xy} = -\nu e^2/h$. According to the Kubo formula, the diagonal term of the conductivity tensor can be expressed entirely in terms of the states at the Fermi surface. The off-diagonal elements of $\hat{\sigma}$ are, on the other hand, determined by all the states below the Fermi level. Therefore, if the Fermi energy is located inside the mobility gap, $\sigma_{xx} = 0$ (and $\rho_{xx} = 0$) at $T = 0$. For $T \neq 0$, the diagonal conductivity is non-zero, but exponentially small. The non-zero contribution is due to activated excitation of the electrons to the extended states belonging to higher Landau levels, or due to variable-range hopping (Ono 1982). The Hall-conductivity σ_H is finite, even at $T = 0$, due to the contribution from extended states below E_F. As the variation of E_F (due to change in n_0 or B) within the mobility gap has no effect on the occupancy of the states (at $T \to 0$), $\sigma_H (= \sigma_{xy})$ remains constant. This accounts for the observation of the plateaus in σ_H as a function of n_0 or B. The accurate quantization of σ_H in the region of a plateau still remained to be explained and is the central issue of several theoretical studies to be discussed below.

3.2. Integral quantization: theoretical work

An explanation of some aspects of the IQHE is possible without introducing the localized states, by invoking the reservoir theory (Baraff and Tsui 1981). According

to this theory, the ionized donors in the heterojunctions serve as a reservoir. The electrons tunnel across a potential barrier to the electron channel. When the density of states in the channel $D(E)$ is a sum of delta-functions, the thermodynamic equilibrium requires that ν is an integer in certain ranges of B. Then, $\sigma_H = \text{const.}$ and so is ν. The reservoir theory does not account for the exact quantization, cannot properly explain the Si-MOSFET system where $D(E)$ is governed by the gate potential and has not received any experimental support (Pudalov et al. 1984).

The theoretical approaches discussed below address the important aspects of the IQHE, viz., the high precision of the QHE and independence of the effects of the sample boundary or the impurities. Also, they shed light on the physical mechanism that is responsible for the occurrence of IQHE.

3.2.1. Kubo formula approach

The Kubo formula (Kubo 1957, Kubo et al. 1965) is a general expression for the current, regarded as a linear response to an external field. In this approach the Hall conductivity is written as

$$\sigma_{xy} = -\frac{n_0 ec}{B} + \Delta\sigma_{xy}, \tag{24}$$

where $\Delta\sigma_{xy}$ is given by the correlation function of the velocities of the center coordinates X and Y as (Aoki and Ando 1981, Ando 1982, 1983b, Aoki 1987)

$$\Delta\sigma_{xy} = \frac{e^2\hbar}{\mathrm{i}A}\sum_{\alpha}\Big\langle f(E_\alpha)\sum_{\beta}\Re(E_\alpha - E_\beta + \mathrm{i}0)^{-2} \times [\langle\alpha|\dot{X}|\beta\rangle\langle\beta|\dot{Y}|\alpha\rangle - \langle\alpha|\dot{Y}|\beta\rangle\langle\beta|\dot{X}|\alpha\rangle]\Big\rangle, \tag{25}$$

where $f(E)$ is the Fermi distribution function, $|\alpha\rangle$ is the eigenstate of the Hamiltonian (16) and A is the area of the system. If $|\alpha\rangle$ is a localized state, then, for any $|\beta\rangle$,

$$\langle\alpha|\dot{X}|\beta\rangle = (\mathrm{i}\hbar)^{-1}\langle\alpha|X|\beta\rangle(E_\alpha - E_\beta). \tag{26}$$

Using eqs. (25) and (26) and the relation $[X, Y] = il_0^2$, the contribution from the state $|\alpha\rangle$ to $\Delta\sigma_{xy}$ in eq. (25) is $\Delta\sigma_{xy}^{\alpha} = f(E_\alpha)ec/B$.

From this result, several important observations (at $T = 0$) can be made:

(a) As long as the Fermi level lies in the energy regime of the localized states, σ_{xy} is constant.

(b) If all the states below the Fermi level E_F are localized, $\sigma_{xy} = 0$ because $\Delta\sigma_{xy}$ exactly cancels* $-n_0 ec/B$. For the QHE to exist, at least one state per Landau level has to be extended. This shows that the presence of a magnetic field provides a situation different from that predicted by the scaling theory (Abrahams et al. 1979). According to that theory, particles moving in a two-dimensional random potential are always localized in the absence of a magnetic field and at $T = 0$.

*A similar result was also obtained by Usov and Ulinich (1982).

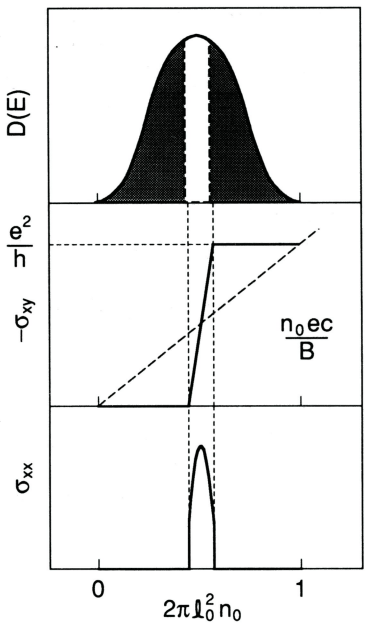

Fig. 6. The diagonal conductivity σ_{xx}, the Hall conductivity σ_{xy} and the density of states $D(E)$ as a function of the Landau-level filling factor.

In the case of a strong magnetic field and when E_F lies in a gap between the nth and $(n+1)$th Landau levels, i.e., electrons occupy states up to the nth Landau level such that $\nu = n$, Aoki and Ando (1981) showed that $\sigma_{xx} = 0$ and $\sigma_{xy} = -ne^2/h$. The situation is depicted in fig. 6 where, at the onset of the upper plateau, $\sigma_{xy} = -e^2/h$ which is what one expects when *all* the states (including localized states) would carry the Hall current.

Prange (1981) explained this apparent paradox by studying a model of free electrons interacting with a δ-function impurity. He concluded that the Hall current at the integral quantization is exactly the same as that for free electrons because the loss of Hall current due to the formation of one localized state is exactly compensated by an appropriate increase of the Hall current carried by the remaining extended states.

There are several other important contributions based on the Kubo formula which should also be mentioned. Thouless (1981) has shown that for the integer quantization, the Hall conductivity (as derived from the Kubo formula) is unaffected by a weak variation of the impurity potential. Streda (1982) reformulated the Kubo approach to write the Hall conductivity as

$$\sigma_H = \sigma_H^I + \sigma_H^{II}, \quad \sigma_H^{II}(E) = -ec \frac{\partial N(E)}{\partial B}\bigg|_{E = E_F}. \tag{27}$$

Here $N(E)$ is the number of states with energy $\leqslant E$, and the first term σ_H^I depends

on the material parameters and impurity potential $U(\boldsymbol{r})$. If, at $T = 0$, the Fermi energy lies in a gap of the energy spectrum of the system, σ_{xx} and $\sigma_{\mathrm{H}}^{\mathrm{I}}$ vanish. If that gap is the nth Landau gap and the degeneracy of the Landau band is the same as that for free-electron Landau levels $N(E_{\mathrm{F}}) = neB/hc$ and $\sigma_{\mathrm{H}}^{\mathrm{II}} = -ne^2/h$. Relation (27) is thermodynamic.

3.2.2. The gauge invariance approach

The universal character of the QHE suggests that the effect is due to a fundamental principle. Laughlin (1981) proposed that the effect is due to the gauge invariance and the existence of a mobility gap. Following Laughlin, we consider a Gedankenexperiment involving the *measurement* of Hall conductivity in the geometry of fig. 7. A two-dimensional electron system is bent into the form of a ribbon. A magnetic field $\boldsymbol{B}$ pierces the ribbon everywhere normal to the surface and a voltage V (Hall voltage) is applied across the edges of the ribbon.

Let us now imagine passing a flux $\boldsymbol{\Phi}$ through a solenoid as shown. The current I is then expressed in terms of the total electronic energy of the system U as (Byers and Yang 1961)

$$I = c\frac{\partial U}{\partial \boldsymbol{\Phi}}. \tag{28}$$

Under a gauge transformation $\boldsymbol{A} \to \boldsymbol{A} + \delta\boldsymbol{A} = \boldsymbol{A} + \delta\boldsymbol{\Phi}/L$, where L is the circumference of the loop, the wave function of the electron acquires a phase factor

$$\psi' \to \psi \exp\left(\mathrm{i}\frac{e}{\hbar c}\frac{\delta\boldsymbol{\Phi}}{L}y\right) = \psi \exp\left(2\pi\mathrm{i}\frac{\delta\boldsymbol{\Phi}}{\boldsymbol{\Phi}_0}\frac{y}{L}\right). \tag{29}$$

If the electron is localized (e.g., trapped by an impurity), the wave function vanishes outside a region of size smaller than L and will not respond to the flux. The energies of localized states are unchanged by the adiabatic process of varying $\boldsymbol{\Phi}$. On the other hand, if the electron is extended, such a transformation is not allowed unless $\delta\boldsymbol{\Phi}/\boldsymbol{\Phi}_0$ is an integer (the wave functions are required to be single-valued). We have seen earlier that the electrons in Landau levels have extended wave functions and will contribute to the current.

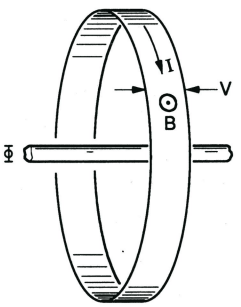

Fig. 7. Hall effect in the geometry of Laughlin's Gedankenexperiment.

Let us now imagine that the magnetic flux through the loop is increased adiabatically from zero. If the Fermi level lies in a mobility gap (i.e., $\sigma_{xx}=0$ as discussed above) any localized states which may be present will not *see* the change. The electrons in the extended states will however, respond to the change until the flux reaches Φ_0, when each of these states must map identically onto themselves (the transformation is unitary). For free electrons, Laughlin (1981) showed that as the flux is increased adiabatically, each state moves to its neighboring state in the direction of the electric field, as in a shift register (Laughlin 1984a). When the flux increases by one quantum, the electron distribution must look exactly the same as before. The net result of the adiabatic process is a transfer of charge from one edge of the loop to the other. If n electrons are transferred during a change $\Delta\Phi$ of one flux quantum, the net change in electronic energy is $\Delta U=-neV$, where V is the potential difference between the edges. Writing $\delta U/\delta\Phi\to\Delta U/\Delta\Phi$, we get $I=-c(neV/\Delta\Phi)=-n(e^2/h)V$, and $\sigma_H=-ne^2/h$. According to Laughlin's approach, the quantization is so accurate because it is based on two very general conditions: the gauge invariance of the system and the existence of a mobility gap.

Laughlin's work has been extended to the case of a dirty system by Giuliani et al. (1983) and also to study the gauge transformation for disordered systems numerical experiments have been performed (Aoki 1982, Aers and MacDonald 1984). Halperin (1982) extended the Laughlin approach to include the role of edge states which will be discussed in § 3.3.3.

3.2.3. The topological invariance approach

We have thus far, considered only the one-electron system and ignored the many-body interaction. The quantization condition in the presence of a many-body interaction was studied by Niu et al. (1985) (see also Avron and Seiler 1985, Aoki and Ando 1986a), who showed that the Hall conductance can be expressed in a topologically invariant form. In the following, we briefly discuss the work of Niu et al. (1985).

The Hall conductivity of a rectangular plane with sides L_1 and L_2 is written via the Kubo formula as (see, e.g., Morandi 1988)

$$\sigma_H=\frac{ie^2\hbar}{A}\sum_{n>0}\frac{(v_1)_{0n}(v_2)_{n0}-(v_2)_{0n}(v_1)_{n0}}{(E_0-E_n)^2}, \tag{30}$$

where $A=L_1L_2$, $|0\rangle$ is the ground state and $|n\rangle$ corresponds to the excited states of the N_e-electron Hamiltonian, E_0 and E_n are the corresponding eigenenergies and the velocity operators v_1 and v_2 are defined as

$$\boldsymbol{v}=\sum_{i=1}^{N_e}\frac{1}{m_i^*}\left(-i\hbar\nabla_i+\frac{e}{c}\boldsymbol{A}_i\right),\quad \boldsymbol{v}=(v_1,v_2). \tag{31}$$

Considering the Landau gauge $\boldsymbol{A}=(0,Bx)$, the wave function ψ for the opposite edges are related by magnetic translations (Brown 1968)

$$\psi(x_i+L_1)=e^{i\alpha L_1}e^{iy_iL_1/l_0^2}\psi(x_i),\quad \psi(y_i+L_2)=e^{i\beta L_2}\psi(y_i), \tag{32}$$

where α and β are phase parameters. Making the unitary transformation

$$\varphi_n=\exp\left(-i\alpha\sum_{i=1}^{N_e}x_i\right)\exp\left(-i\beta\sum_{i=1}^{N_e}y_i\right)\psi_n,$$

one can write the Hall conductivity as (Niu et al. 1985)

$$\sigma_H = \frac{ie^2}{\hbar}\left(\left\langle \frac{\partial\varphi_0}{\partial\theta}\middle|\frac{\partial\varphi_0}{\partial\varphi}\right\rangle - \left\langle \frac{\partial\varphi_0}{\partial\varphi}\middle|\frac{\partial\varphi_0}{\partial\theta}\right\rangle\right), \tag{33}$$

where $\theta = \alpha L_1$ and $\varphi = \beta L_2$.

One major condition for quantization in this approach is that the Hall conductivity is a *local* response function, insensitive to the boundary condition. We can, therefore, average over all the phases ($0 \leqslant \theta < 2\pi, 0 < \varphi \leqslant 2\pi$) that specify different boundary conditions

$$\sigma_H = \bar{\sigma} = \frac{e^2}{h}\int_0^{2\pi} d\theta \int_0^{2\pi} \frac{d\varphi}{2\pi i}\left(\left\langle \frac{\partial\varphi_0}{\partial\varphi}\middle|\frac{\partial\varphi_0}{\partial\theta}\right\rangle - \left\langle \frac{\partial\varphi_0}{\partial\theta}\middle|\frac{\partial\varphi_0}{\partial\varphi}\right\rangle\right). \tag{34}$$

This can be evaluated as

$$\sigma_H = \frac{e^2}{h}\int_0^{2\pi} d\theta \int_0^{2\pi} \frac{d\varphi}{2\pi i}\,\nabla \times \boldsymbol{A} = \frac{e^2}{h}\frac{1}{2\pi i}\oint d\boldsymbol{l}\cdot\boldsymbol{A}. \tag{35}$$

The vector $\boldsymbol{A}$ is defined by its components

$$A_\gamma = \frac{1}{2}\left(\left\langle \frac{\partial\varphi_0}{\partial\gamma}\middle|\varphi_0\right\rangle - \left\langle \varphi_0\middle|\frac{\partial\varphi_0}{\partial\gamma}\right\rangle\right), \quad \gamma = \theta, \varphi. \tag{36}$$

Now, if the ground state is nondegenerate and is separated from the excited states by a gap, the ground state can only change by a phase factor depending on θ and φ. For example, the ground state must go back to itself (up to an overall phase factor) as θ and φ change by 2π. Therefore, $\oint d\boldsymbol{l}\cdot\boldsymbol{A} = 2\pi i \times$ (integer) and $\sigma_H =$ integer $\times\, e^2/h$.

Integral (34) is a topological invariant and was originally obtained for noninteracting electrons in a two-dimensional periodic potential (Thouless et al. 1982). An excellent introduction to the connection between the IQHE and the topological idea can be found in the book by Morandi (1988). Similar arguments for the FQHE have also been put forward by Niu et al. (1985). A better treatment of the latter case was given, however, by Tao and Haldane (1986) and will be discussed briefly in § 4.2.6.

3.3. Other developments

3.3.1. Electron localization in the quantum Hall regime

We have seen above that the extended and localized states form an integral part of our understanding of the IQHE. Several authors have studied the role of various disorder potentials in the IQHE regime (Iordansky 1982, Kazarinov and Luryi 1982, Luryi and Kazarinov 1983, Trugman 1983, Joynt and Prange 1984, Apenko and Lozovik 1985).

For an external potential $U(\boldsymbol{r})$ which varies slowly over length scales of order l_0, and neglecting the inter-Landau-level transition, the motion of an electron reduces to a drift of the center of cyclotron motion along an equipotential line $U(\boldsymbol{r}) = \text{const}$. The state of an electron with energy E (measured from the center of the Landau band) is localized near the line $U(\boldsymbol{r}) = E$, and decreases exponentially from this line. The properties of the equipotential lines for a two-dimensional random potential are fairly well known from the classical percolation model (Zallen and Scher 1971). If the potential $U(\boldsymbol{r})$ is symmetric about $E = 0$ (e.g., for equal number of attractive and repulsive scatterers), the equipotentials with $E \neq 0$ are closed curves of finite length, and correspond to localized states. If we call the regions with $U(\boldsymbol{r}) > E$ land and those with $U(\boldsymbol{r}) < E$ water, we get a set of islands for $E > 0$ or a number of lakes for $E < 0$. In that case, we have at least one coast line at $E = 0$. In this picture, all the states in a two-dimensional system subjected to a strong magnetic field are localized, except for the state at the center of the Landau band, which is extended. In an external electric field, analyzing the delocalized drift trajectories, Iordansky (1982) and others (Kazarinov and Luryi 1982, Luryi and Kazarinov 1983, Trugman 1983) obtained the quantization for σ_{xy}.

The behavior of the localization length in strong magnetic fields has been a very active area of research. Using diagrammatic techniques, Ono (1982) was the first to suggest that only the center of the Landau level is extended while the other states are exponentially localized. Ando studied the electron localization by numerically diagonalizing the Hamiltonian matrix for a system with short-range δ-function (Ando 1983a) and long-range (Gaussian form) impurities (Ando 1984) which are randomly distributed. The localization length was studied by Ando via the Thouless-number method. The Thouless number $g(L)$ is defined as the ratio of the shifts ΔE of the individual energy levels due to a change in boundary conditions (from periodic to anti-periodic) to the level separation $[L^2 D(E)]^{-1}$, where $D(E)$ is the density of states per unit area. For localized states, one can determine the extent of the localized wave functions or the inverse localization length $\alpha(E)$, from

$$g(L) = g(0)\, \mathrm{e}^{-\alpha(E)L},$$

where L is the sample length (Licciardello and Thouless 1978). The conclusion was that the states are exponentially localized except in the vicinity of the center of each Landau band. The inverse localization length was found to decrease almost linearly with energy when the energy is far from the center of the Landau band, and then smoothly approaches zero with energy. Extremely large localization lengths in the center of the Landau band were also found in numerical studies by other groups (Schweitzer et al. 1984). Finite-size studies with sample size $\sim 50\,000$ times the cyclotron radius (Aoki and Ando 1985, 1986b, Ando and Aoki 1985) resulted in $\alpha(E) \propto |E - E_n|^s$ near the center E_n, of each Landau level with the critical exponent $s \lesssim 2$ and $s \lesssim 4$ for $n = 0$ and $n = 1$, respectively. From a perturbative calculation, Hikami (1986) estimated $s = 1.9 \pm 0.2$. The critical exponent from the classical percolation model is $\frac{4}{3}$ (Trugman 1983) and including quantum tunneling one gets $s = \frac{7}{3}$ (Mil'nikov and Sokolov 1988). A percolation model, including quantum tunneling and interference near the percolation threshold (Chalker and Coddington 1988)

yielded a one-parameter scaling behavior and a critical exponent $s = 2.5 \pm 0.5$. Recent finite-size scaling studies in the lowest Landau band (Huckestein 1990, Huckestein and Kramer 1990) revealed a universal one-parameter scaling behavior and a critical exponent $s = 2.34 \pm 0.04$. For more recent results on the finite-size scaling, see the paper by Huo and Bhatt (1992).

3.3.2. Renormalization group approach

This approach is a result of the attempts to unify the weak localization in a two-dimensional electron system (Abrahams et al. 1979, Gorkov et al. 1979) and the quantization of the Hall conductivity. According to the two-parameter scaling theory of the IQHE (Pruisken 1984, 1985, Levine et al. 1983, Khmel'nitzkii 1983, 1984), σ_{xx} and σ_{xy} (in units of e^2/h) vary with a length scale L, given by the renormalization group equations

$$\frac{d\sigma_{\eta\nu}}{d\xi} = \beta_{\eta\nu}(\sigma_{xx}, \sigma_{xy}), \quad \xi = \ln L, \tag{37}$$

where $\beta_{\eta\nu}$ is a periodic function of σ_{xy} with a period of e^2/h. The resulting phase diagram is shown in fig. 8. The beginning of each flow line is selected at a point corresponding to a spatial scale $\xi = 0$, where $\hat{\sigma}$ can be estimated from the classical formula (21). With increasing system size, all flow lines merge into one of the fixed points at $(\sigma_{xx}, \sigma_{xy}) = (0, -ne^2/h)$ which are the *localization fixed points* and describe localized wave functions of the electrons near the Fermi energy E_F. In addition, the system also has unstable fixed points (denoted by $\otimes$) where the flow lines with $\sigma_{xy} = -(n + \frac{1}{2})e^2/h$ terminate. At these points, $\sigma_{xx} > 0$ and describes the singular behavior in the renormalized transport coefficients, corresponding microscopically to the occurrence of a diverging localization length. These *delocalization fixed points* are associated with the extended states at E_F. It should be noted that the explicit form of the β-function is not known, only its asymptote is calculated. An approximate

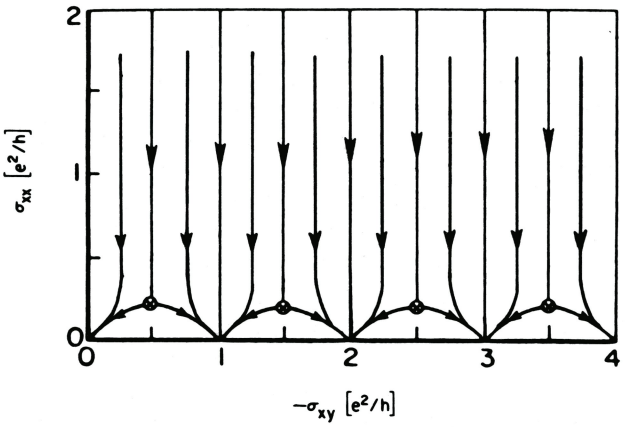

Fig. 8. Scaling diagram for the integral quantum Hall effect.

calculation of σ_{xx} at the unstable fixed points gives $\sigma_{xx} = (1.4/\pi)e^2/h$ for a white-noise random potential (Hikami 1984).

The predictions of the two-parameter scaling theory have been tested experimentally (Wei et al. 1985b, 1986). The study was carried out on the 2DES in InGaAs/InP heterostructures. It was assumed that the effective sample size is governed by inelastic scattering which can be varied via the temperature T, and a phase diagram was constructed (fig. 9) from the measurements of $\sigma_{xx}(T)$ and $\sigma_{xy}(T)$. In this figure, the dashed lines are from the temperature range 4.2–10 K, where lowering of T is accompanied by the usual enhancement of the Shubnikov–de Haas oscillations as a result of a change of the Fermi distribution. In the scaling region ($T \sim 0.5$–4.2 K) the results, presented as full lines, show a tendency (as $T \to 0$) to flow out toward the fixed points $(0, e^2/h)$, $(0, 2e^2/h)$, $(0, 3e^2/h)$ and $(0, 4e^2/h)$. Detailed studies by these authors indicated that there is a remarkable symmetry about the line $\sigma_{xy} = 1\frac{1}{2}$. The data are consistent with the existence of an intermediate coupling, delocalization critical point. A phase diagram for the FQHE based on the scaling hypothesis has also been proposed (Laughlin et al. 1985). The theories are still in a state where a fair amount of *guesswork* is required as input.

The two-parameter scaling theory of the IQHE has not received much support from other theoretical work. Finite-size studies of the diagonal and off-diagonal conductivities (Ando 1986) indicate that they are *not* independent but possess single flow lines dependent on the Landau-level index which contradicts the scaling theory results discussed above.

3.3.3. Current-carrying edge states

In calculating the Hall current in § 3.1.2, we assumed that the current is distributed over the entire surface of the two-dimensional layer. Some authors have pointed out

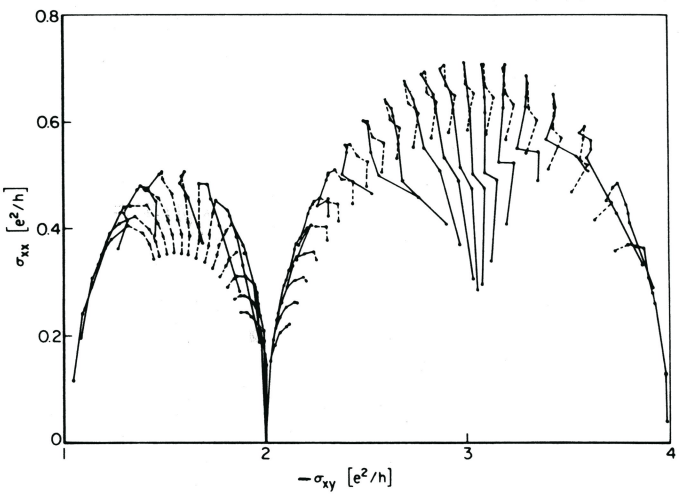

Fig. 9. Experimental results for the conductivity plotted as T-driven flow lines from $T = 10$ to 0.5 K. The dashed lines are from 10 to 4.2 K and the solid lines from 4.2 to 0.5 K.

that, in fact, the current flows along the one-dimensional edge of the layer (Halperin 1982, Thouless 1982, MacDonald and Streda 1984).

Let us consider the noninteracting electrons of § 2.2 but now confined in a potential

$$\begin{aligned} V(x) &= 0, \qquad x \in (-\tfrac{1}{2}L_x, \tfrac{1}{2}L_x) \\ &= +\infty, \quad \text{otherwise.} \end{aligned} \tag{38}$$

Far from the boundary inside the sample, $V(x) \equiv 0$ and the eigenstates are as derived above. However, near the boundary, the energy eigenvalues deviate from the Landau energy (7) and behave as in fig. 10, as a function of X. The total current carried by the nth branch of the energy spectrum when all the bulk states for that branch are occupied is given by (MacDonald and Streda 1984)

$$I_n = \frac{e}{h}(\mu^{\mathrm{R}} - \mu^{\mathrm{L}}) \tag{39}$$

where μ^{R} is the chemical potential at the right (positive-x) edge of the sample and μ^{L} is the chemical potential at the left edge.

In the equilibrium case, $\mu^{\mathrm{R}} = \mu^{\mathrm{L}}$, the bulk current which flows in the presence of an electric field (in which case the above result also holds) is exactly cancelled by the surface diamagnetic current. For noninteracting electrons all the current would flow at the edges and the quantized Hall current would be just the difference between the two edge currents. MacDonald and Streda (1984) also showed that these results are unaltered by the inclusion of the random potential in the Hamiltonian (Apenko and Lozovik 1985). The role of edge currents has also been studied by Smrcka (1984), who derived the Hall current from the Kubo formula, taking into account the edge states explicitly. He concluded that the IQHE is exclusively due to the edge currents. It has been proposed that the edge currents might be observed from the oscillations in the magnetic susceptibility (Azbel 1985). Several experiments exploring the current-carrying edge states have been reported in the literature (van Wees et al. 1989, Müller et al. 1990).

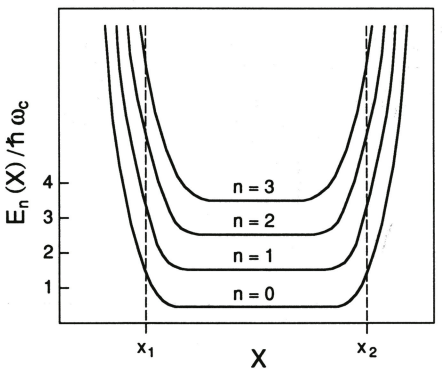

Fig. 10. Energy spectrum of a noninteracting electron in an infinite well confining potential.

3.3.4. *Further topics*

Finally, we should also mention the work on the IQHE based on the Landauer (1970) approach to transport theory in a two-dimensional system subjected to a strong perpendicular magnetic field. In this approach the conductivity is related to the transmission and reflection probabilities of electrons which are incident from the leads. Streda et al. (1987) studied the transport in the case where the current is incident from a disorder-free (ideal) lead on the left of the sample and is collected by an ideal lead on the right. They argued that the magnetoresistance R and the Hall resistance R_H satisfy the sum rule $(R + R_H)^{-1} = (e^2/h)\,\mathrm{Tr}(t^\dagger t)$, where t is the transmission matrix. Büttiker (1988) has proposed a theory of the IQHE in open multiprobe conductors based on the formation of the edge states and the suppression of backscattering (from one side of the sample to the other). Experiments on the transport through narrow channels (Haug et al. 1989, Washburn et al. 1988) have revealed interesting results but the literature on transport in narrow channels is too numerous to cover in this general review and the interested reader is referred to an excellent review by Beenakker and van Houten (1991).

The IQHE breaks down at a critical current density j_c (the corresponding critical Hall field $E_H = 60$ V/cm at a magnetic field $B = 5$ T) where the resistivity increases abruptly by orders of magnitude and the Hall plateau disappears (Ebert et al. 1983, Sakaki et al. 1984, Pudalov and Semenchinsky 1984a). A typical result is shown in fig. 11. At a current density of $j_x = j_c = 0.5$ A/m, ρ_{xx} at the center of the $\nu = 2$ plateau increases dramatically. A pronounced hysteresis is observed near the breakdown threshold.

In the literature several mechanisms are proposed as possible causes for the breakdown.

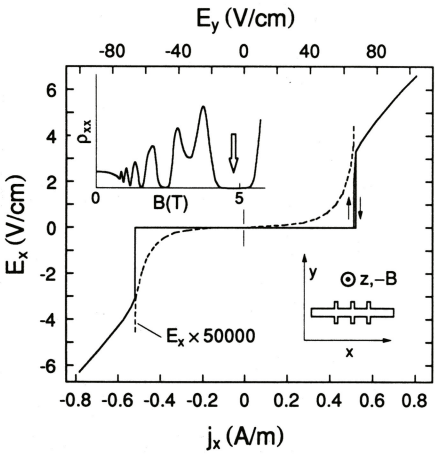

Fig. 11. Breakdown of the IQHE under quantum field effect conditions. The current–voltage characteristic of a GaAs-heterostructure at a filling factor $\nu = 2$. The device geometry and the $\rho_{xx}(B)$ curve are shown as insets.

(a) Transition between levels accompanied by the absorption or emission of phonons (Streda and von Klitzing 1984).

(b) Zener effect: tunneling of the carriers between occupied and empty Landau levels (Cage et al. 1983).

(c) Heating instability associated with the breakdown of the balance between the energy gained and released by electrons (Ebert et al. 1983, Komiyama et al. 1985).

4. Fractional quantum Hall effect

The FQHE was realized in the high-mobility 2DEG in GaAs/$Al_xGa_{1-x}As$ heterostructures (Tsui et al. 1982, Störmer et al. 1983b, Störmer 1984, Tsui and Störmer 1986) and in high mobility Si-MOSFETs (Pudalov and Semenchinsky 1984b). It is characterized by the fact that the Hall conductance has plateaus quantized to certain simple *fractions* ν of the unit e^2/h and at the same place, the longitudinal resistivity shows an almost dissipationless current flow. Here ν is a rational fraction with an odd denominator [in fact, an even denominator fraction has also been found to exhibit the FQHE (Clark et al. 1986, Willett et al. 1987)].

Figure 12 shows the effect for a few values of the filling factor ν. For $\nu > 1$, the characteristic features of the integral QHE are clearly visible. However, in the extreme quantum limit, i.e., for $\nu < 1$, and at low temperatures ($T = 0.48$ K), one observes a clear minimum in ρ_{xx} and a quantized Hall plateau at $\nu = \frac{1}{3}$. In later experiments

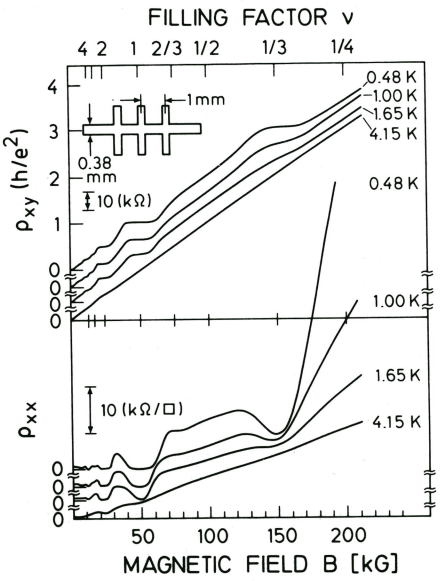

Fig. 12. The Hall resistivity ρ_{xy} and the diagonal resistivity ρ_{xx} as a function of magnetic field in a GaAs-heterostructure with electron density $n_0 = 1.23 \times 10^{11}$ cm^{-2}, where the FQHE was first discovered. The scale on the top shows the *filling factor*, defined in the text.

with higher mobility samples the plateau was found to be quantized to better than three parts in 10^5 and ρ_{xx} was lower than $0.1\Omega/\square$. A weak structure around $\nu = \frac{2}{3}$ is also visible at the lowest temperatures. Subsequent experiments with higher mobility samples (Chang et al. 1984, Willett et al. 1987, Sajoto et al. 1990) revealed several other fractions which are displayed in fig. 13 and summarized below:

$$\nu = \tfrac{1}{3}, \tfrac{1}{5}, \tfrac{1}{7}*, \tfrac{2}{5}, \tfrac{2}{7}, \tfrac{2}{9}*, \tfrac{2}{11}*, \tfrac{3}{7}, \tfrac{3}{11}*, \tfrac{3}{13}*, \tfrac{3}{17}*, \tfrac{4}{9}, \tfrac{4}{11}*, \tfrac{4}{13}*, \tfrac{5}{11}, \tfrac{6}{13}*, \tfrac{7}{15}*$$
$$= \tfrac{2}{3}, \tfrac{4}{5}, \tfrac{3}{5}, \tfrac{5}{7}, \tfrac{4}{7}, \tfrac{5}{9}*, \tfrac{6}{11}*, \tfrac{7}{13}*$$
$$= \tfrac{4}{3}, \tfrac{7}{5}, \tfrac{9}{7}, \tfrac{5}{3}, \tfrac{8}{5}, \tfrac{10}{7}, \tfrac{13}{9}*, \tfrac{11}{7}, \tfrac{7}{3}*, \tfrac{8}{3}*, \tfrac{5}{2}.$$

The fractions with an asterisk (*) have as yet shown structures in ρ_{xx} only. The first row of fractions are simply $\nu = p/q\ (2p < q)$. The second row contains the fractions $\nu = 1 - p/q$ and the last row contains the other fractions $\nu = 1 + p/q$, $1 + (1 - p/q)$, $2 + p/q$, etc. The arrangements of the fractions are made with the *electron–hole symmetry* in mind which will be described below.

The major characteristics of FQHE, viz., the appearance of plateaus in ρ_{xy} and minima in ρ_{xx}, are similar in nature to those of the IQHE. In spite of this apparent similarity the physical origin of the former effect is different from the latter as can be seen from the fact that:

(a) plateaus and minima appear at *fractional* filling factors where no structure is expected in the single-electron density of states, and

(b) the effect is observed only in samples of very high mobility.

This leads us naturally to believe that the electron–electron interaction plays a major role in the FQHE. The main theoretical step in explaining the FQHE, therefore, is to determine the properties of an interacting 2DEG with a neutralizing background

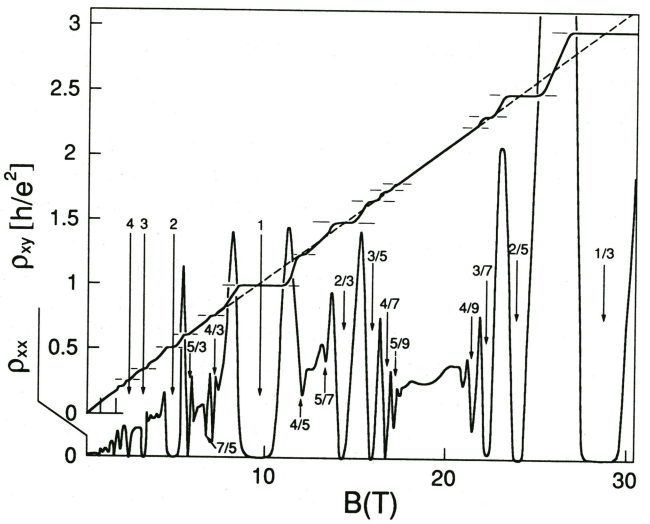

Fig. 13. Overview of the observed fractions in the FQHE measurements.

subjected to a strong perpendicular magnetic field such that only the lowest Landau level is partially filled.

In our attempt to explain the fractional Hall steps, we have to understand the exceptionally stable states of the electron system at particular rational values of ν. The pinning of the density at those values of ν would require that the energy versus density curve shows a cusp-type behavior. A cusp would imply a discontinuity of chemical potential, which would in turn, mean that the electron system is, in fact, *incompressible** at those stable states.

4.1. Ground state

As the fractional Hall steps are observable only in samples of very high mobility, impurity potentials are not expected to be very important in comparison with electron–electron interactions. The first step in the explanation of the FQHE, as stated above, would, therefore, be to study the properties of a system of two-dimensional interacting electrons in a uniform positive background with the magnetic field strength such that only the lowest Landau level is partially filled. One could then consider the effect of impurities as a perturbation. The unit of potential energy is $e^2/\varepsilon l_0$, which is taken to be the energy scale throughout. Here, ε is the background dielectric constant. For a magnetic field $B \gtrsim 10$ T, where the FQHE is generally observed, using the values $\varepsilon \approx 12.9$ and $m^* \approx 0.067 m_e$ which are appropriate for GaAs, it is easy to verify that $e^2/\varepsilon l_0 \lesssim \hbar\omega_c$, the cyclotron energy. The admixture of states in higher Landau levels can thus be safely ignored as a first approximation.

In the extreme quantum limit (no Landau-level mixing), the Hamiltonian describing the two-dimensional electron gas can be written as

$$\mathscr{H} = \tfrac{1}{2} \sum_q V(q)[\bar{\rho}(q)\,\bar{\rho}(-q) - \rho e^{-q^2 l_0^2}], \tag{40}$$

where $\bar{\rho}(q)$ is the projected density operator (discussed in § 4.3.1.) and ρ is the average density of the particles in the system. In the absence of kinetic energy, the ground state is expected to be a solid – which means that the Hamiltonian describes a set of interacting classical particles. The operators $\bar{\rho}(q)$ do not commute with one another. However, in the limit $B \to \infty$ we get (Fukuyama et al. 1979), $[\bar{\rho}_q, \bar{\rho}_{-q}] \to l_0$ and one obtains a (presumably triangular) solid. The earlier attempts to explain the FQHE were mostly centered on crystal-state calculations. Such calculations (Yoshioka and Lee 1983, Yoshioka et al. 1983) however, did not find any singularity at $\nu = \frac{1}{3}$. It is also difficult conceptually to understand how the crystal could carry electric current with no resistive loss, since the charge-density wave (CDW) would be pinned by impurities. Also, if the CDW is not allowed to move with a drift velocity $\boldsymbol{E} \times \boldsymbol{B}c/B^2$, the electron contribution to the Hall conductivity would vanish (Laughlin 1987).

The earliest numerical calculation of the ground state including Coulomb interactions, was by Yoshioka et al. (1983). They investigated the eigenstates of an electron system in a periodic rectangular geometry, by numerically diagonalizing the

* The compressibility $\kappa \sim [\mathrm{d}^2 E/\mathrm{d}\nu^2]^{-1} \to 0$ at the point where the cusp appears.

Hamiltonian. Their results, as we shall discuss in the following section, revealed several interesting features; the most important result was, of course, that the ground state has a significantly lower energy than that of a Hartree–Fock (HF) Wigner crystal.

4.1.1. Theoretical work

In the lowest Landau level, the single-particle wave function is (except for the normalization factor),

$$\varphi = \exp[\mathrm{i}Xy/l_0^2 - (X-x)^2/2l_0^2], \tag{41}$$

where $X = k_y l_0^2$ is the center coordinate of the cyclotron motion. Following Yoshioka et al. (1983), let us now put a finite number of electrons in the cell and introduce interactions among them. Consider the situation where there are a few electrons in a rectangular cell of sides a and b, and a strong magnetic field perpendicular to the x–y plane. The electrons are considered to be in the lowest Landau level and are spin-polarized. Applying periodic boundary conditions in the y-direction, one obtains $k_y = X_j/l_0^2 = 2\pi j/b$ for an integer j. For the periodic boundary condition along the x-direction, let us write, for an integer m, $X_m = a$. Clearly, $ab = 2\pi l_0^2 m$, and from eq. (13), m is the Landau-level degeneracy N_s. For N_e electrons in the cell, the filling factor is easily calculated as $\nu = 2\pi l_0^2 N_e/ab = N_e/N_s$. The Hamiltonian for the finite-electron system, suitably modified for the periodic rectangular geometry, is diagonalized numerically to obtain the eigenstates. Details of the calculation can be found in the literature (Yoshioka et al. 1983, Yoshioka 1984a, Chakraborty and Pietiläinen 1988a).

In fig. 14, we present the numerical results of Yoshioka et al. (1983), for the ground-state energies per particle as a function of filling fraction in the lowest Landau level. There are several interesting features noticeable in the result. Let us first consider the case $\nu = \frac{1}{3}$. The ground-state energy per particle for four-, five- and six-electron systems are extremely insensitive to the system size. They are also very close to the infinite system result to be discussed below.

The ground-state energies tend to have downward *cusps* for $\nu = \frac{1}{3}$ and $\frac{2}{5}$. (A cusp is also visible at $\nu = \frac{1}{2}$ but the ground-state energies at this filling fraction strongly depend on the system size.) Cusps at the experimentally observed filling fractions are quite naturally required, in order to describe the incompressible fluid state first proposed by Laughlin.

In fig. 14, also presented are the energy of the crystal state for the $N_e = 4$ system obtained in the exact diagonalization (closed squares) and HF approximation (open squares) (Yoshioka et al. 1983). The long- and short-dashed lines show the energy of the electron and hole crystals resulting from the HF approximation for the infinite system. These crystal state results show a smooth behavior at $\nu = \frac{1}{3}$ and are not the lowest energy state.

In the lowest Landau level, there is electron–hole symmetry. The system with N_e electrons in N_s sites is thus equivalent to that with $(N_s - N_e)$ holes in N_s sites. When we choose the products of the single-electron eigenstates as a basis, the off-diagonal matrix elements for $\nu = N_e/N_s$ are the same as those for $\nu = (1 - N_e/N_s)$ filling

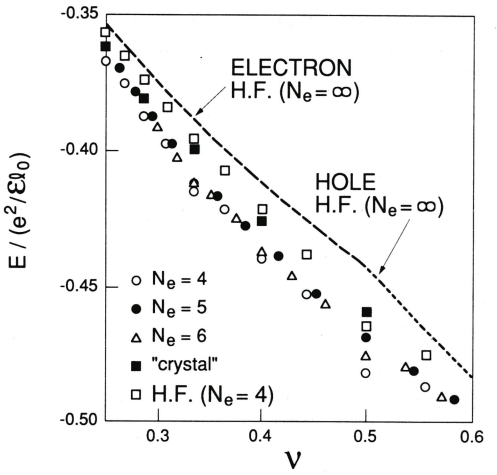

Fig. 14. Energies per particle for finite-electron systems in a periodic rectangular geometry, as a function of filling factor in the lowest Landau level. The long- and short-dashed lines are the energy of the electron and hole crystals within the HF approximation for the infinite system. Closed squares denote the crystal state energies for the $N_e = 4$ system. Open squares show the energy of the crystal state for the $N_e = 4$ system obtained in the HF approximation.

fractions for the same J values. The diagonal matrix elements differ only by a constant. The results for $\nu > \frac{1}{2}$, shown in fig. 14, are obtained by the above arguments.

While these numerical results demonstrate that the ground state is not crystalline, as Halperin (1983) pointed out, not much insight is gained about the ground-state wave function. The crucial step in obtaining a ground-state wave function for a translationally invariant liquid state, and the mechanism for stabilizing the system at particular densities, was made by Laughlin (1983a, 1984b, 1987), and is described below. It should be pointed out that similar finite-size calculations were also done by Haldane and Rezayi (1985) in a spherical geometry.

In discussing Laughlin's theory, we will closely follow the seminal paper by Laughlin (1983a) as well as papers by other authors (Halperin 1983, Chakraborty and Pietiläinen 1988a). Let us consider the electrons to be confined in the x–y plane and subjected to a magnetic field perpendicular to the plane. Considering the symmetric gauge vector potential, $\boldsymbol{A} = \frac{1}{2}B(x\hat{y} - y\hat{x})$, it is convenient to regard the x–y plane as a complex plane. For the lowest Landau level, the single-particle wave functions are eigenfunctions of orbital angular momentum

$$\varphi_m(z) \equiv |m\rangle = \frac{1}{(2\pi l_0^2 2^m m!)^{1/2}} \left(\frac{z}{l_0}\right)^m \mathrm{e}^{-|z|^2/4l_0^2}, \tag{42}$$

with $z = x - \mathrm{i}y$ being the electron position. The many-electron system is described by a Hamiltonian of the form

$$\mathscr{H} = \sum_j \left[\frac{1}{2m_e} \left| -\mathrm{i}\hbar\nabla_j - \frac{e}{c}\boldsymbol{A}_j \right|^2 + V(z_j) \right] + \sum_{j<k} \frac{e^2}{|z_j - z_k|}, \tag{43}$$

where z_j is the location of the jth electron and $V(z)$ is the potential generated by a

uniform neutralizing background. It is easy to verify (Laughlin 1983b) that

$$\langle m|r^2|m\rangle = 2(m+1)l_0^2, \tag{44}$$

which means that the area covered by a single electron in state $|m\rangle$ moving in its cyclotron orbit is proportional to m. This result might be considered as an indication of the relation between the interelectron spacing and the angular momentum. From eq. (13), it is readily noticeable that the degeneracy of a Landau level N_s is the upper bound to the quantum number m. This is seen by requiring that, $\pi\langle r^2\rangle \leqslant A$, where A is the area of the system. One then obtains from the above relations, $m \leqslant N_s - 1$, where N_s is the Landau-level degeneracy. Therefore, the state space of an electron in the lowest Landau level is spanned by $1, z, z^2, \ldots, z^{N_s-1}$ times the exponential factor $e^{-|z|^2/4l_0^2}$.

The *Jastrow-type* many-electron wave function proposed by Laughlin for the $\nu = 1/m$ state is

$$\psi_m = \prod_{\substack{j,k=1 \\ j<k}}^{N_e} (z_j - z_k)^m \prod_{j=1}^{N_e} e^{-|z_j|^2/4l_0^2}. \tag{45}$$

For m being an odd integer, this wave function obeys Fermi statistics. The wave function is entirely made up out of states in the lowest Landau level. It is also an eigenstate of the angular momentum with eigenvalue $M = \frac{1}{2}N_e(N_e - 1)m$. The total angular momentum M is the degree of the polynomial (conservation of angular momentum). In order to gain more insight about the wave function, let us expand the first product in powers of z_1, keeping all other coordinates fixed. The highest power would then be $m(N_e - 1)$, which must be equal to $N_s - 1$. For large N_e, we then obtain

$$m \cong \frac{A}{2\pi l_0^2 N_e} = \frac{1}{\nu}. \tag{46}$$

The parameter m is thus fixed by the density and, unlike in conventional Jastrow theory, we do not have a variational parameter in the trial wave function. For $m = 1$ (filled Landau level), the polynomial $\Pi_{j<k}(z_j - z_k)$ is the Vandermonde determinant of order N_e. As $N_e \to \infty$, the particle density in this state tends to $1/2\pi l_0^2$ (Bychkov et al. 1981).

For $m > 1$, the wave function vanishes as a high power of the two-electron separation, and thus tends to minimize the repulsive interaction energy. The probability distribution of the electron for ψ_m is given by

$$\begin{aligned} |\psi_m|^2 &= e^{-\mathcal{H}_m}, \\ \mathcal{H}_m &= -2m \sum_{j<k} \ln|z_j - z_k| + \sum_j |z_j|^2/2l_0^2. \end{aligned} \tag{47}$$

This is the Hamiltonian for a *charge neutral* two-dimensional classical plasma where the particles are interacting via a two-dimensional Coulomb (logarithmic) interaction with each other and with a uniform neutralizing background with particle density $\rho_m = 1/2\pi l_0^2 m$. Therefore, in order to achieve charge neutrality, the plasma particles spread out uniformly in a disk with density ρ_m, corresponding to a filling factor $\nu =$

$1/m$, and m is an *odd* integer. The classical plasma provides strong support that the Laughlin state is indeed a translationally invariant liquid (Caillol et al. 1982).

The electrons however, in contrast to the classical plasma particles, interact via the *three-dimensional* Coulomb interaction, and the expectation value of the potential energy in a quantum state is given by

$$\frac{\langle V \rangle}{N_e} = \tfrac{1}{2} \int v(r)\, g(r)\, \mathrm{d}\boldsymbol{r}, \tag{48}$$

where $g(r)$ is the two-particle radial distribution function, which was calculated by Laughlin using the classical plasma approach. In this calculation, Laughlin used the hypernetted-chain (HNC) theory, which is a well-established technique for dealing with the classical plasma and other quantum systems (Chakraborty and Pietiläinen 1988a). The pair-correlation function in the Laughlin state shows a predominantly short-range behavior: for small r, $g(r)$ goes to zero as r^{2m} whereas for the crystal state it goes as r^2.

Immediately after Laughlin introduced his ground-state wave function (45), Halperin (1983) gave a very interesting interpretation of the significance of this wave function. Imagine freezing the positions of all but the first particles and let χ be the wave function for z_1, parametrized by the positions of the other particles:

$$\chi(z_1) = \prod_{j=1}^{L} (z_1 - Z_j) \exp(-\tfrac{1}{4}|z_1|^2). \tag{49}$$

From the theory of analytic functions one knows that the polynomial can be uniquely defined by the set of its zeros $\{Z_j;\, j = 1, \ldots, L\}$. In fact, the zeros of the wave function (45) look like two-dimensional Coulomb charges (in a plasma analogy) which repel the particle at z_1. These zeros must be uniformly distributed with density $1/2\pi$ in order to balance the effect of the Gaussian term which is attracting the particles into the origin. Halperin then observed that Laughlin's wave function places these zeros directly on the other particles. Each particle sees m zeros bound to the positions of the other particles. Let us define a *vortex* as a point where the wave function is zero, such that the phase changes by 2π as one circulates around the vortex in the negative direction. The special feature of the Laughlin wave function is that there are exactly m vortices at each electron position and no other *wasted* vortices elsewhere in the plane.

The ground-state energy for the $\nu = \frac{1}{3}$ state, obtained in the HNC scheme is $E(\nu = \frac{1}{3}) = -0.4056 e^2/\varepsilon l_0$. This result is quite close to the exact results shown for finite electron systems in the earlier section. There have also been very accurate Monte Carlo studies to confirm these results on quantitative grounds. Levesque et al. (1984) obtained the ground-state energy of the Laughlin wave function by evaluating the pair-correlation functions for about 256 particles, using the method described by Caillol et al. (1982). For the ground-state energy they obtained $E(\nu = \frac{1}{3}) = -0.4100 \pm 0.0001$ and $E(\nu = \frac{1}{5}) = -0.3277 \pm 0.0002$ (in units of $e^2/\varepsilon l_0$).

4.1.2. Recent developments

The spectroscopic method, based on radiative recombination of two-dimensional electrons with photoexcited holes, has generated considerable interest recently as a route to study the FQHE. Magneto-optical experiments in the FQHE regime are expected to reveal physics of the many-electron state that is not accessible in magneto-transport. While transport measurements study the electronic properties close to the Fermi energy, the method of radiative recombination provides information about the whole energy spectrum of the electrons.

Buhmann et al. (1990) studied the spectra of the radiative recombination of two-dimensional electrons in several GaAs–AlGaAs heterostructures with photoexcited holes localized in a δ-doped monolayer of acceptors (Be atoms) 25 nm away from the interface. They observed anomalies in the luminescence spectrum at $\nu = \frac{2}{3}, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}$, and $\frac{1}{9}$. This was the first observation of the filling factor $\frac{1}{9}$.

Goldberg et al. (1990) also reported an interesting experiment which involved simultaneous measurement of the transport resistivity components and photoluminescence spectra from one-side-doped GaAs/AlGaAs single quantum wells. They observed a sudden shift of the luminescence peak and a minimum in peak intensity at $\nu = \frac{2}{3}$. Precisely at $\nu = \frac{2}{3}$ the peak position was found to shift abruptly to the blue by ~ 0.1 meV. In the high-field region about the FQHE states $\nu = \frac{2}{5}$ and $\frac{1}{3}$, Goldberg et al. observed a splitting of the luminescence. Anomalies in the energy of photoluminescence near $\nu = \frac{2}{3}$ were also observed by Turberfield et al. (1990). A popular account of these studies can be found in the article by Worlock (1990). Some theoretical work on optical recombination in the FQHE regime has appeared in the literature (Bychkov and Rashba 1989, Apal'kov and Rashba 1991, Chakraborty and Pietiläinen 1991).

At very low electron density, i.e., for very small ν, the ground state is expected to be a Wigner crystal. It should be noted that the two-dimensional one-component classical plasma has a crystallization transition which occurs at $m \simeq 70$ (Caillol et al. 1982). Therefore, it would be interesting to estimate the filling factor at which the liquid to solid transition takes place for the present quantum system. Most computations of the crystal energy have been within the HF approximation (Yoshioka and Lee 1983, Yoshioka et al. 1983). Comparing the accurate Monte Carlo results for the Laughlin liquid state with the HF crystal energies as a function of filling factor, the critical filling was found to occur at $\nu_c \sim \frac{1}{10}$ (Levesque et al. 1984), an estimate similar to that of Laughlin (1983a). With a variational wave function for a *correlated* Wigner crystal the crossover point was estimated by Lam and Girvin (1984) to be $\nu_c^{-1} = 6.5 \pm 0.5$.

A clear indication of the FQHE at $\nu = \frac{1}{5}$ has been observed in the experiments of Mallett et al. (1988), who determined the activation energy of this state [$\Delta \sim 50$ mK at 19 T] for the first time (see § 4.2.4 for a discussion on the energy gap). They also reported observation of weak ρ_{xx} minima at $\nu = \frac{2}{9}$, and $\frac{2}{11}$, thereby providing experimental confirmation of the $\frac{1}{5}$ hierarchy, which is different from the sequence of states obtained from $\frac{1}{3}$. Furthermore, the observation of the ρ_{xx} minimum at $\frac{2}{11}$ sets a lower limit to which the FQHE is shown to persist.

The difficulty of performing measurements in the low-ν, low-temperature range was greatly circumvented recently by Goldman et al. (1988). In recent magnetotransport measurements they observed a structure near the filling factor $\nu = \frac{1}{7}$. It was interpreted as evidence for a developing fractional quantum Hall state. The Wigner crystallization is, therefore, expected to occur for $\nu < \frac{1}{7}$. A structure in ρ_{xy}, indicating the FQHE state at $\nu = \frac{1}{7}$ in a magnetic field up to 18 T, was also observed by Wakabayashi et al. (1988). In a recent experiment, Jiang et al. (1990) also found that at filling factors below $\frac{1}{5}$ as well as in a narrow region above $\nu = \frac{1}{5}$, ρ_{xx} diverges exponentially as $T \to 0$. This is interpreted as indicative of a pinned solid-like phase in the underlying electronic system. This electronic phase is reentrant in a narrow region above $\nu \sim \frac{1}{5}$. The liquid state at $\nu = \frac{1}{5}$ is apparently embedded within a solid phase.

A two-dimensional electron solid has also been observed in very high-mobility Si-MOSFETs at large filling factors ($\nu > 1$) and low magnetic fields ($B < 5$ T) (D'Iorio et al. 1990, Kravchenko et al. 1992). Below a carrier concentration of $\approx 10^{11}$ cm^{-2}, the longitudinal resistance maxima near the filling factors 1.5 and 2.5 increases sharply by more than four orders of magnitude. The magnitude of the resistance maxima is thermally activated below 0.5 K and the activation energy at 8×10^{10} cm^{-2} is about 1 K. In the range of concentrations and filling factors where the electron solid is observed, the current–voltage characteristics show a sharp threshold at ≈ 100 mV/cm (D'Iorio et al. 1992).

Spectroscopic measurements in the FQHE regime have been recently employed to probe the Wigner crystallization in the strongly interacting electron system at low filling factors. A qualitative phase diagram $T_c(\nu)$ proposed by Buhmann et al. (1991) is shown in fig. 15 where the incompressible liquid state occurs at as low a density as $\nu = \frac{1}{9}$ but for $\nu < \frac{1}{5}$ all these states are surrounded by Wigner-crystal-like states (shaded regions).

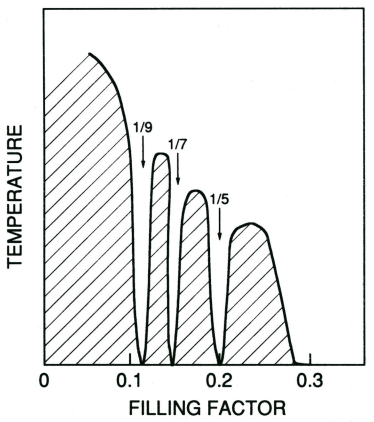

Fig. 15. Qualitative form of the phase diagram $T_c(\nu)$. Regions above the solid curve represent the incompressible liquid state. Shaded regions represent a solid phase.

4.2. Elementary excitations

One important result of Laughlin's theory was the observation that the elementary charged excitations in a stable state $\nu = 1/m$ are quasiparticles and quasiholes with *fractional* electron charge of $e^* = \pm e/m$. If one electron is added to the system, it amounts to adding m elementary excitations and, hence, the discontinuity in slope of the energy curve can be written as

$$\left.\frac{\partial E}{\partial N_e}\right|_{\nu_+} - \left.\frac{\partial E}{\partial N_e}\right|_{\nu_-} = m(\tilde{\varepsilon}_p + \tilde{\varepsilon}_h) = mE_g, \tag{50}$$

where E_g is the energy required to create one quasiparticle (with energy $\tilde{\varepsilon}_p$) and one quasihole (with energy $\tilde{\varepsilon}_h$) well separated from each other. The pinning of the density to $\nu = \frac{1}{3}$ suggests that there will be no low-frequency phonon-type excitations at long wavelengths. When the filling factor ν is slightly shifted from the stable state $1/m$, with m being an odd integer, the ground state of the system is expected to consist of a small density of quasiparticles or quasiholes, with charge $\pm e/m$ and Coulomb interactions. In the presence of impurities, these quasiparticles or quasiholes are expected, for low concentrations, to be trapped in potential fluctuations.

4.2.1. Quasiholes and quasiparticles

For densities slightly different from $\nu = 1/m$, we cannot construct a wave function with exactly m vortices tied to each electron. In order to have an electron density slightly *less* than $1/m$, either we add a few extra vortices not tied to electron positions, or have some electrons with more than m vortices (recall Halperin's interpretation of the Laughlin's wave function discussed above). Laughlin considered the first choice, and the wave function is written (in units where $l_0 = 1$) as

$$\psi_m^{(-)} = \exp\left(-\tfrac{1}{4}\sum_l |z_l|^2\right)\prod_j (z_j - z_0)\prod_{j<k}(z_j - z_k)^m, \tag{51}$$

where $z_0 = x_0 - iy_0$. This wave function has a simple zero at $z_j = z_0$ for any j, as well as m-fold zeros at each point where $z_j = z_k$, for $k \neq j$. Writing

$$|\psi_m^{(-)}|^2 = e^{-\mathscr{H}_m^{(-)}}, \tag{52}$$

we obtain

$$\mathscr{H}_m^{(-)} = \mathscr{H}_m + 2\sum_j \ln|z_j - z_0|, \tag{53}$$

which is just the Hamiltonian of a classical one-component plasma in the presence of an extra repulsive *phantom* point charge at point z_0, whose strength is less, by a factor $1/m$, than the charges in the plasma. The plasma will neutralize this phantom by a *deficit* of $1/m$ charge near z_0, while elsewhere in the interior of the system, the charge density will not be changed. However, the real three-dimensional electric charge is carried by the electrons and by the uniform positive background, and *not* by the phantom. As the electron charge density cancels the uniform background, a

real net charge $-e/m$ is accumulated in the vicinity of z_0. The wave function (51), therefore, describes a *quasihole* at point z_0.

In the case where the electron density is slightly *higher* than the stable $1/m$ state, the choice of the wave function is not so clear. In this case, one clearly needs a state with one flux quantum (or equivalently, one zero of the wave function) missing. The wave function proposed by Laughlin for the quasiparticle state is written

$$\psi = \prod_{j=1}^{N_e} \left[e^{-|z_j|^2/4l_0^2} \left(2l_0^2 \frac{\partial}{\partial z_j} - z_0^* \right) \right] \prod_{l<k} (z_l - z_k)^m. \tag{54}$$

In this case, the square of the wave function is not directly interpretable as the distribution in a classical statistical-mechanics problem. However, Laughlin (1984a,b) has provided a means to calculate the charge density. Laughlin (1984a,b) (and later Chakraborty 1985) used a two-component HNC scheme to calculate the energy cost associated with the creation of the elementary excitations. The energies evaluated via the Monte Carlo scheme show a large discrepancy with the HNC results (Morf and Halperin 1986). For details, see Chakraborty and Pietiläinen (1988a).

The quasiparticle and quasihole size is the distance over which the one-component plasma screens. For a weakly coupled plasma this distance is the Debye length (Hansen and Levesque 1981), $\lambda_D = l_0/\sqrt{2}$. For the strongly coupled plasma relevant in the present case, a better estimate is the ion-disk radius associated with *charge* $1/m$, $R = \sqrt{2}\, l_0$. In this sense, the quasiparticles have the same *size* as electrons in the lowest Landau level.

4.2.2. Quasiparticle statistics

The fractional charge of the quasiparticles and quasiholes was derived by Laughlin from the plasma analogy discussed above. Later, Arovas et al. (1984) presented a direct method to determine the charge of excitations of the Laughlin state. Interestingly, that approach also revealed some surprising features about the *statistics* of the quasiparticles and quasiholes.

If the quasihole of charge e^* is carried around a loop enclosing the flux $\boldsymbol{\Phi}$ it will gain a change in phase of

$$\gamma = \frac{e^*}{\hbar c} \oint \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{l} = 2\pi \frac{e^*}{e} \frac{\boldsymbol{\Phi}}{\boldsymbol{\Phi}_0}. \tag{55}$$

On the other hand, if we let the quasihole position z_0 move adiabatically around this loop, the rate of this change can be calculated from

$$\frac{\mathrm{d}\gamma(t)}{\mathrm{d}t} = \mathrm{i} \left\langle \psi_m^{(-)} \middle| \frac{\mathrm{d}\psi_m^{(-)}}{\mathrm{d}t} \right\rangle.$$

When we now substitute the time derivative

$$\frac{\mathrm{d}\psi_m^{(-)}}{\mathrm{d}t} = \sum_i \frac{\mathrm{d}}{\mathrm{d}t} \ln[z_i - z_0(t)] \psi_m^{(-)} \tag{56}$$

from eq. (51), we get

$$\frac{\mathrm{d}\gamma}{\mathrm{d}t} = \mathrm{i}\left\langle \psi_m^{(-)} \middle| \frac{\mathrm{d}}{\mathrm{d}t} \sum_i \ln(z_i - z_0) \middle| \psi_m^{(-)} \right\rangle. \tag{57}$$

Introducing the one-particle density

$$\rho^{(-)}(z) = \left\langle \psi_m^{(-)} \middle| \sum_i \delta(z_i - z) \middle| \psi_m^{(-)} \right\rangle \tag{58}$$

of the quasihole state, this can be written as

$$\frac{\mathrm{d}\gamma}{\mathrm{d}t} = \mathrm{i} \int \mathrm{d}x\, \mathrm{d}y\, \rho^{(-)}(z) \frac{\mathrm{d}}{\mathrm{d}t} \ln[z - z_0(t)], \tag{59}$$

where $z = x + \mathrm{i}y$. We now expand $\rho^{(-)}(z)$ around the uniform density $\rho_0 = \nu \boldsymbol{\Phi}/\boldsymbol{\Phi}_0$ of the ground state, i.e., we write $\rho^{(-)}(z) = \rho_0 + \delta\rho^{(-)}(z)$. Let us first consider the ρ_0 term. When we integrate z_0 around a circle of radius R the total variation of $\ln[z - z_0(t)]$ will be $2\pi\mathrm{i}$ if $|z| < R$ and zero if $|z| > R$. Thus, the corresponding change of phase is given by

$$\begin{aligned} \gamma_0 &= \mathrm{i} \int_{|r|<R} \mathrm{d}x\, \mathrm{d}y\, \rho_0 2\pi\mathrm{i} \\ &= -2\pi\langle n\rangle_R = -2\pi\nu \frac{\boldsymbol{\Phi}}{\boldsymbol{\Phi}_0}, \end{aligned} \tag{60}$$

where $\langle n\rangle_R$ is the mean number of electrons in a disk of radius R. The correction $\delta\rho^{(-)}(z)$ is localized in the vicinity of z_0, so its contribution to the change of phase is of the order of size of the quasihole, i.e., of the order of l_0^2. In comparison with the constant term contribution, which is of the order R^2, this can be clearly neglected. From eqs. (55) and (60) we can now deduce the charge of the quasihole to be $e^* = -ve$, in agreement with Laughlin's theory. Similar arguments can also be made for the quasiparticles.

Let us now consider two quasiholes located at z_a and z_b, a distance $|z_a - z_b| = R$ apart. When we carry the hole z_b adiabatically around the circle of radius R centered at the hole z_a the analysis presented above is still valid provided the mean electron number $\langle n\rangle_R$ in the disk is counted correctly. We find that the quasihole at z_a has a charge $-ve$ which we can interpret to mean that exactly ν electrons are removed from the disk, so we must substitute $\langle n\rangle_R - \nu$ for the mean electron number. Hence, an extra phase of amount $\Delta\gamma = 2\pi\nu$ is gained in the process. The interchange of two quasiparticles can be achieved by letting each of them make a turn of π around the other. From the treatment above it is obvious that the accumulation of this extra phase is continuous and, therefore, the total change of phase will be $\nu\pi$. For $\nu = 1$, the quasiholes behave like fermions. For fractional fillings, however, we end up with the striking conclusion that the quasiholes (and quasiparticles) obey neither Bose nor Fermi statistics, but *fractional* statistics. The concept of anomalous statistics in two dimensions was first introduced by Leinaas and Myrheim (1977). Indistinguishable quantum particles in two-dimensional space can, in general, have anomalous

statistics. This is in contrast to three-dimensional space (and spaces of higher dimensionality), where there are only Bose and Fermi statistics. According to Wilczek (1982), particles obeying anomalous statistics are called *anyons* because the interchange of any two of them can result in *any* phase.

4.2.3. *Hierarchies of quasiparticles*

Thus far we have discussed the theoretical work on the FQHE aiming at an explanation of the behavior of the electron system at $\nu = 1/m$ (and its electron–hole symmetric fractions). But as shown in fig. 13, the FQHE appears at several higher order filling fractions, notably, $\frac{2}{5}, \frac{3}{7}, \frac{4}{9}$, etc. In order to explain the FQHE at these fractions, Laughlin's theory needs to be modified.

According to Haldane (1983), there is a hierarchical system in which p/q states with $1 < p < q$ are formed from new generation of elementary excitations in the same manner as the Laughlin state is formed by the electrons. Each new generation of elementary excitations appears against a *vacuum* formed by the preceding generation. In this picture, the elementary excitations are supposed to obey Bose statistics, unlike the electrons which obey Fermi statistics. The chain of equations are

$$n_s = mN_L + \alpha_1 N^{(1)}, \tag{61a}$$

$$N_L = p_1 N^{(1)} + \alpha_2 N^{(2)}, \tag{61b}$$

$$N^{(1)} = p_2 N^{(2)} + \alpha_3 N^{(3)}, \tag{61c}$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots$$

$$N^{(k-1)} = p_k N^{(k)}; \tag{61d}$$

where $m \geqslant 1$ is odd if all $p_j \neq 0$ are even and $\alpha_j = \pm 1$ and $n_s = 1/2\pi l_0^2$ is the degeneracy per unit area of the lowest Landau level. The above equations are understood as follows: in the first equation the imbalance between the electron density N_L and the density of the incompressible state n_s/m is compensated by the $N^{(1)}$ particles ($\alpha_1 = -1$) or holes ($\alpha_1 = +1$). The next equation is the same except that it describes the next generation with m replaced by an even quantity p_1. The solution of eqs. (61) gives the continued fraction for $\nu = N_L/n_s$:

$$\nu = \cfrac{1}{m + \cfrac{\alpha_1}{p_1 - \cfrac{\alpha_2}{p_2 - \cfrac{\alpha_3}{p_3 - \ldots}}}} \quad (\nu < 1). \tag{62}$$

A new liquid $[m, \alpha_1 p_1, \ldots, \alpha_j p_j]$ does not form before the appearance of the preceding liquid $[m, \alpha_1 p_1, \ldots, \alpha_{j-1} p_{j-1}]$.

The hierarchical scheme of Halperin (1984) is very much in the spirit of Laughlin's theory. In this scheme, the quasiparticles are required to obey *fractional statistics*. If ν_t is a stable filling factor obtained at level t of the hierarchy, the low-lying energy

states for filling factors near to ν_t can be described by the addition of a small density of quasiparticle excitations to the ground state at ν_t. There are two types of elementary excitations: p-excitations (particle like) and h-excitations (hole like), with charges $q_t e$ and $-q_t e$, respectively. Halperin then constructed a *pseudo*-wave function (where the coordinates are for quasiparticles),

$$\psi(Z_1, ..., Z_{N_t}) = P[Z_j]\, Q[Z_j] \exp\left(-\frac{|q_t|}{4l_0^2} \sum_{j=1}^{N_t} |Z_j|^2 \right), \tag{63}$$

where Z_j is the complex coordinate of the quasiparticle j and N_t is the number of quasiparticles. The polynomial $P[Z_j]$ is chosen to be *symmetric*,

$$P[Z_j] = \prod_{i<j} (Z_i - Z_j)^{2p_{t+1}}, \tag{64}$$

where p_{t+1} is a positive integer, whose variational properties are known from Laughlin's theory. The function $Q[Z_j]$ determines the symmetry properties of ψ under the interchange of quasiparticles. As mentioned above, in contrast to three-dimensional space (and spaces of higher dimensionality), indistinguishable particles in the two-dimensional space can, in general, have anomalous statistics. The bound system of a spinless particle of charge q and an infinitesimally thin solenoid carrying a flux $\boldsymbol{\Phi}$ and cutting through the plane of the motion of the particle [the system was called an *anyon* by Wilczek (1982)] is such an example. The many-body wave function of N_t identical anyons is (Wu 1984a,b):

$$\psi^*(\boldsymbol{r}_1, ..., \boldsymbol{r}_{N_t}) = \prod_{i<j} \exp\left(\mathrm{i} \frac{\xi}{\pi} \varphi_{ij} \right) \psi(\boldsymbol{r}_1, ..., \boldsymbol{r}_{N_t}). \tag{65}$$

In eq. (65), φ_{ij} is the azimuthal angle of the relative vector $(\boldsymbol{r}_i - \boldsymbol{r}_j)$ and $\psi(\boldsymbol{r}_1, ..., \boldsymbol{r}_{N_t})$ is a single-valued wave function. According to eq. (65), a permutation of any two anyons ($\Delta\varphi_{ij} = \pi$) multiplies ψ^* by the phase factor $e^{i\xi}$. Therefore, when $\eta = \xi/2\pi = q\boldsymbol{\Phi}/2\boldsymbol{\Phi}_0$ is an integer, the anyons are bosons and when η is a half-integer, they are fermions. Here $\boldsymbol{\Phi}$ is the usual magnetic flux.

In terms of the complex coordinates Z_j and Z_j^*, the multivalued wave function (65) takes the form

$$\psi^* = \prod_{i<j} (Z_i - Z_j)^{\xi/\pi} f(Z_i, Z_j^*), \tag{66}$$

where $f(Z_i, Z_j^*) = (r_{ij})^{-\xi/\pi} \psi(Z_j, Z_j^*)$ is a single-valued function of the coordinates.

Let us now consider the first step of the hierarchy when the anyon quasiparticles are derived from electrons. The phase change $\Delta\gamma$ of the wave function of a quasiparticle in a closed adiabatic path around another quasiparticle is $\Delta\gamma = 2\pi q$ and for $\nu = 1/m$, $|\Delta\gamma| = 2\pi/m$. Therefore, the change of phase that accompanies the interchange of quasiparticles is $\frac{1}{2}\Delta\gamma = \pi\nu = \xi$ (see § 4.2.2). According to the general structure of eq. (66), the factor $Q[Z_j]$ is now

$$Q[Z_j] = \prod_{i<j} (Z_i - Z_j)^{-\alpha/m_t}. \tag{67}$$

In eq. (67), $\alpha = \pm 1$, according to whether one is dealing with particle- or hole-type excitations, and m_t is a rational number $\geqslant 1$.

Assuming that at any stage of the hierarchy, the quasiparticles or quasiholes can be treated as point particles with pairwise Coulomb interactions, an estimate for the potential energy can be obtained. The resulting energy versus density curve was generated by Halperin for various filling fractions as shown in fig. 16. In this figure, the stable fractions appear with downward pointing cusps.

Finally, the hierarchical scheme described above has been employed by Zhang and Chakraborty (1986) to study the condensation of the spin-reversed quasiparticles.

4.2.4. Energy gap

We have seen in § 4.1.1 that the ground-state energy per electron in a finite-size calculation shows a cusp-like behavior at $\nu = \frac{1}{3}$. As has already been discussed above, the appearance of a cusp means a positive discontinuity in the chemical potential (Halperin 1986) [see eq. (50)]:

$$E_g = \tfrac{1}{3}(\mu_+ - \mu_-), \tag{68}$$

where the chemical potential is defined as

$$\mu = E_0(\nu) + \nu \frac{dE_0}{d\nu}, \tag{69}$$

and E_0 is the ground-state energy per particle. In eq. (68), $\mu_\pm$ corresponds to $\nu_\pm = N_e/(N_s \mp 1)$. The factor $\frac{1}{3}$ corresponds to the fractional charge of the quasiparticles discussed above.

The finite-size system result for the energy gap by Chakraborty et al. (1986) and Chakraborty and Pietiläinen 1986) is shown in fig. 17. Extrapolation of the results for spin-polarized three- to seven-electron systems (plotted as solid circles) leads to (approximately) $E_g \simeq 0.1 e^2/\varepsilon l_0$. These studies also indicate that in the absence of the Zeeman energy, spin-reversed excitations cost even less energy (fig. 17) than the pure Coulomb excitations proposed by Laughlin (1983a), and will be discussed below.

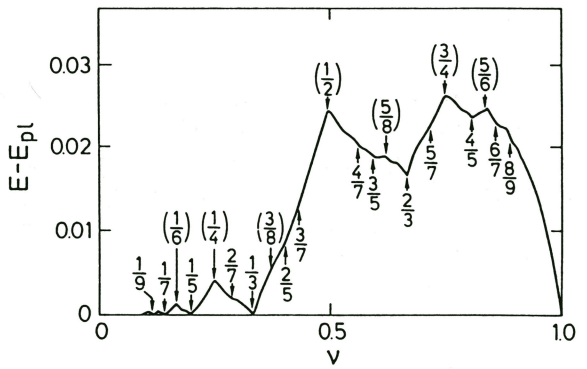

Fig. 16. Potential energy $E(\nu)$ per quantum of magnetic flux (in units of $e^2/\varepsilon l_0$) versus the filling factor ν of the lowest Landau level. The plasma energy E_{pl} has been subtracted at each filling fraction.

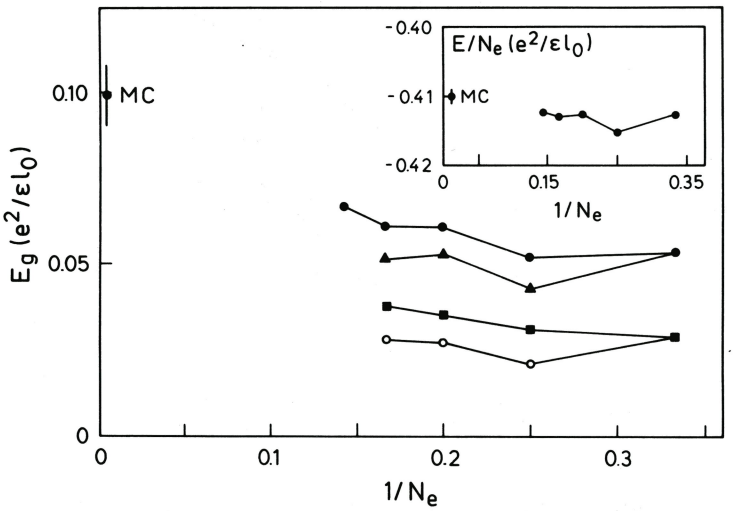

Fig. 17. The energy gap for different spin polarization of the quasiparticle (q.p.) and quasihole (q.h.) excitations for three- to seven-electron systems, ignoring the Zeeman energy. In the case of spin-aligned q.p. + q.h. gap (●), Monte Carlo (MC) results are from Morf and Halperin (1986). The other three cases are: spin-aligned q.p. + spin-reversed q.h. (▲); spin-reversed q.p. + spin-aligned q.h. (■); and spin-reversed q.p. + q.h. (○). The ground-state energies (Yoshioka et al. 1983) at $\nu = \frac{1}{3}$ are given in the inset for comparison.

In both the integer and fractional QHE, the vanishing of the diagonal resistivity implies a gap in the excitation spectrum. In the case of the IQHE, the gap is in the single-particle density of states, whereas in the FQHE, the gap lies in the excitation spectrum of the correlated many-electron ground state. The energy gap is usually obtained from the temperature dependence of the magnetoconductivity, σ_{xx} [or ρ_{xx} since near the ρ_{xx} minima, $\rho_{xx} \ll \rho_{xy}$, and $\sigma_{xx} = \rho_{xx}/(\rho_{xx}^2 + \rho_{xy}^2) \sim \rho_{xx}/\rho_{xy}^2$], as $\sigma_{xx} \propto \rho_{xx} \propto \exp(-W/k_B T)$, where $W = \frac{1}{2}E_g$ is the activation energy and k_B is the Boltzmann constant. In the case of the IQHE, Tausendfreund and von Klitzing (1984) found the energy gap obtained from activation energy measurements to be close to the cyclotron energy. For the FQHE, similar measurements for the energy gap have been undertaken by several experimental groups.

A systematic study of the energy gap for the filling fractions, $\nu = \frac{1}{3}, \frac{2}{3}, \frac{4}{3}$, and $\frac{5}{3}$ was reported by Boebinger et al. (1985, 1987). The activation energies for these filling factors are presented in fig. 18. The following features are noteworthy in these results:

(i) No apparent sample dependence was observed.

(ii) The data for $\nu = \frac{1}{3}$ (solid symbols with error bars) and $\frac{2}{3}$ (open symbols) overlap at $B \sim 20$ T. The data for $\nu = \frac{4}{3}$ and $\frac{5}{3}$ (■ at 5.9 and 7.4 T) are consistent with the data for $\nu = \frac{2}{3}$ at similar magnetic fields. Collectively, the experimental results therefore suggest a *single* activation energy (or *gap* E_g), for all the filling fractions mentioned above.

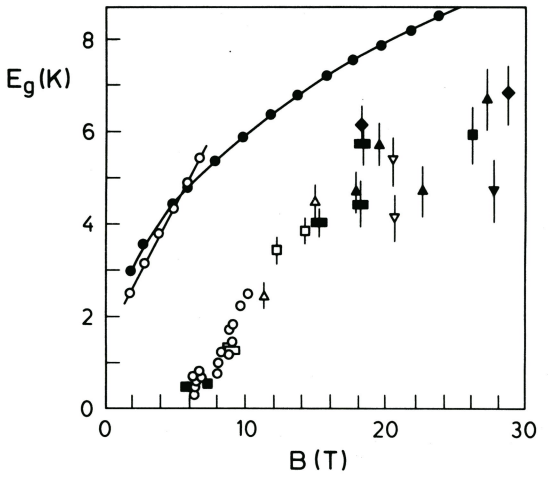

Fig. 18. Activation energies for the $p/3$ fractions (in units of K) as a function of magnetic field. Also shown are the theoretical results (solid lines) for the energy E_g (K) versus magnetic field B (Tesla) for a five-electron system. The open points on the theoretical curve (Chakraborty et al. 1986, Chakraborty and Pietiläinen 1986) are for spin-reversed quasiparticles and spin-polarized quasiholes, while the filled points are for the fully spin-polarized case.

(iii) E_g does not follow the expected $B^{1/2}$ dependence. For $B \lesssim 5.5$ T, E_g is vanishingly small, but at higher magnetic fields, there is a roughly *linear* increase in E_g up to $\sim$12 T.

In our discussions of the electron system thus far, we have ignored the finite spread of the electron wave function perpendicular to the two-dimensional plane. In the real systems, the wave function of an electron has a finite spread in the z-direction. It is well known that inclusion of the finite-thickness correction effectively softens the short-range divergence of the bare Coulomb interaction, when the interelectron spacing is comparable with the inversion layer width (MacDonald and Aers 1984, Zhang and Das Sarma 1986). This causes a reduction in the energy gap calculated for the ideal system. At low magnetic fields, mixing of the higher Landau levels also influences the energy gap somewhat (Yoshioka 1984b). The activation energy for $p/3$ states has also been measured by Willett et al. (1988) in an ultrahigh-mobility ($\mu = 5 \times 10^6$ cm^2 V^{-1} s^{-1}) quasi-two-dimensional electron system in GaAs–(AlGa)As, for which the finite thickness of the carrier system is accurately known. The data are in remarkably good agreement with the theoretical results for the energy gap when Landau-level mixing and the finite-thickness effect are taken into account.

Finally, from the measurement of the activation energy, Clark et al. (1988) made a very important observation. They reported a systematic study of $\sigma_{xx}^c = \sigma_{xx}(1/T = 0)$:

$$\sigma_{xx}^c = \frac{\rho_{xx}^c}{(\rho_{xx}^c)^2 + \rho_{xy}^2} = \frac{\rho_{xx}^c}{(\rho_{xx}^c)^2 + \left[\left(\frac{q}{p}\right)\frac{h}{e^2}\right]^2}$$

which is valid at $\nu = p/q$ and ρ_{xx}^{c} is defined by $\rho_{xx} = \rho_{xx}^{c} e^{-E_g/kT}$. Clark et al. found that, within experimental errors, σ_{xx}^{c} is constant for p/q fractions of the same q. In fig. 19, the conductivity data σ_{xx} are plotted as a function of T^{-1} at both integer and fractional ν. Straight-line fits to the linear region, which corresponds to thermal activation of the quasiparticles and quasiholes across the energy gap, are shown and extrapolated to $T^{-1} = 0$. The extrapolated σ_{xx} intercept is close to e^2/h for the IQHE, whereas for $\nu = p/q$, $\sigma_{xx}^{c} \simeq (1/h)(e/q)^2 = e^{*2}/h$, which is consistent with Laughlin's prediction of the fractional charge of the quasiparticles and quasiholes discussed above.

4.2.5. Spin-reversed quasiparticles

In most of the theoretical work a major assumption was that, because of the strong magnetic fields involved, only one spin state is present. Halperin (1983) first pointed out that in GaAs, the electron g-factor is one-quarter of the free electron value. Therefore, the Zeeman energy is approximately sixty times smaller than the cyclotron energy. It might then be possible to have some electrons with *reversed* spins when the magnetic field is not too large. In the case when one half of the electrons have spins antiparallel to the field, Halperin constructed a simple Laughlin-type state of the form

$$\psi = \prod_{i<j} (z_i - z_j)^m \prod_{\alpha<\beta} (z_\alpha - z_\beta)^m \prod_{i,\alpha} (z_i - z_\alpha)^n \prod_i e^{-|z_i|^2/4l_0^2} \prod_\alpha e^{-|z_\alpha|^2/4l_0^2}, \tag{70}$$

where Roman and Greek indices correspond to spin-up and spin-down electrons,

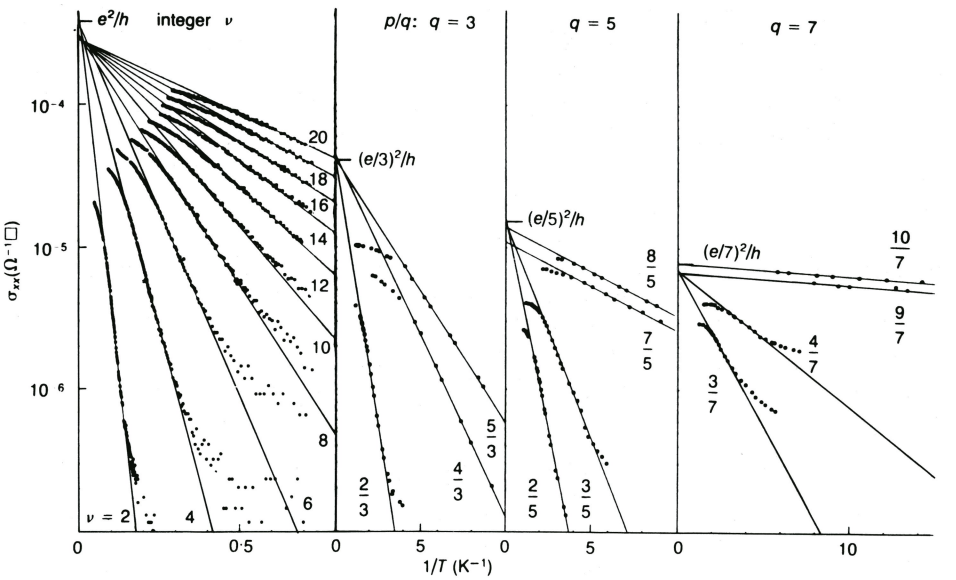

Fig. 19. Conductivity σ_{xx} versus T^{-1} results which measure the quasiparticle charge $e^* = e/q$ at the integer and fractional filling factors $\nu = p/q$.

respectively. Using the classical plasma approach, the filling factor is given by $\nu = 2/(m+n)$. Halperin suggested a study of the particular case of $m=3$, $n=2$ ($\nu=\frac{2}{5}$). The wave function then has the following desirable properties:

(a) all electrons are in the lowest Landau level, so the kinetic energy is an absolute minimum,

(b) electrons of the same spin are kept apart very well because the wave function vanishes as the cube of their separation; electrons of opposite spin are kept apart only slightly less well because the wave function vanishes as the square of the separation, and

(c) the wave function is antisymmetric under interchange of two electrons of the same spin, as required.

The wave function can also be shown to give a total spin $S=0$ state.

For $\nu=\frac{2}{5}$, the wave function given above was studied by Chakraborty and Zhang (1984) to examine the importance of unpolarized spin and it was concluded that, at a very low magnetic field, a spin-unpolarized state for $\nu=\frac{2}{5}$ would be energetically favored. In a later investigation, a systematic study of spin reversal in various filling fractions was attempted by these authors (Zhang and Chakraborty 1984). The finite-size calculations described in § 4.1.1, were employed for this purpose and were suitably generalized for various spin polarizations in a straightforward manner. The energy spectrum for the Hamiltonian is classified in terms of the total spin S and its z-component S_z. For a given S, the spectrum is identical for different values of S_z.

In table 1, we present the results for a finite electron system for the case of a polarized state ($S=2$), a partly polarized state ($S=1$) and the unpolarized state ($S=0$). As seen in table 1, except for $\nu=\frac{1}{3}$, the unpolarized states are energetically favored, as compared to the fully polarized state. For $\nu=\frac{1}{3}$, the spin state $S=2$ is found to be energetically favored compared with the other spin states, even in the absence of Zeeman energy. This result is quite supportive of Laughlin's state at $\nu=\frac{1}{3}$, which is fully antisymmetric. For other filling fractions, the possibility exists that the energy could be lowered by introducing the spin degree of freedom.

Table 1
Potential energy (per particle) for various filling fractions and spin polarization. The Zeeman energy is not included in the energy values which are in units of $e^2/\varepsilon l_0$.

ν	Potential energy				Ground state
	$S=0$	$S=1$	$S=2$	$S=3$	
$\frac{1}{3}$	−0.4135	−0.4120	−0.4152	–	Polarized
$\frac{2}{3}$	−0.5331	−0.5291	−0.5257	−0.5232	Unpolarized
$\frac{2}{5}$	−0.4464	−0.4410	−0.4403	–	Unpolarized
$\frac{2}{7}$	−0.3884	−0.3868	−0.3870	–	Unpolarized
$\frac{3}{5}$	−0.5074	−0.5096	−0.5044	−0.50104	Partially polarized
$\frac{4}{9}$	−0.4554	−0.4600	−0.4528	–	Partially polarized
$\frac{4}{11}$	−0.4241	−0.4278	−0.4219	–	Partially polarized
$\frac{4}{13}$	−0.3970	−0.3997	−0.3975	–	Partially polarized

In the preceding section, we discussed the results for the energy gap E_g from finite-size calculations in a periodic rectangular geometry and noticed that spin-reversal for the quasiparticles cost less energy (in the absence of the Zeeman energy) compared to fully spin-polarized quasiparticles and quasiholes. In the Laughlin approach, a spin-reversed quasiparticle is best visualized in the following wave function for the quasiparticles (Morf and Halperin 1986, Chakraborty 1986):

$$\psi = \prod_{j=2}^{N_e} (z_j - z_1)^{-1} \psi_m, \tag{71}$$

where ψ_m is the ground state given by eq. (45). Bringing in the plasma analogy one can show that $|\psi|^2$ is the distribution function for a two-dimensional plasma in which particle 1 has its charge *reduced* by a factor $1-(1/m)$ in its repulsive interaction with the other particles. Particle 1 has, however, the same interaction as the other particles in its attractive interaction with the background. Particle 1 will, therefore, be attracted to the center of the disk, while a two-dimensional *bubble* will be formed near the origin of size $1-(1/m)$. As a result, there will be an extra negative charge e/m near the origin. Furthermore, when eq. (71) is considered to be a function of the position of any electron other than the singled-out electron 1, there will be one less zero of the wave function than in the case of the ground-state ψ_m.

The magnetic field dependence of E_g for a five-electron system is shown in fig. 17, where we have presented only the lowest energy results. For low magnetic fields, the curve for the lowest energy excitations rises *linearly* as a result of the *spin-reversed quasiparticles*, which include the dominant contribution from the Zeeman energy, itself linear in magnetic field. As the magnetic field is increased, a crossover point is reached, beyond which the $B^{1/2}$ dependence (somewhat modified due to the finite-thickness correction) is then obtained due to the spin-polarized quasiparticles and quasiholes (Chakraborty et al. 1986, Chakraborty and Pietiläinen 1986). These theoretical predictions were made several years ago, the experimental evidence for the presence of spin-reversed quasiparticles has been reported only recently and will be discussed below.

The first clear-cut evidence for the existence of spin-unpolarized states in the ground state of several filling factors was from the experiments by the Oxford group (Clark et al. 1989, Clark and Maksym 1989) where the magnetic field was tilted from the direction perpendicular to the electron plane. It is known that the exchange and correlation contributions to the ground-state energy depend to a first approximation on $B_\perp$ alone. The Zeeman energy, on the other hand, depends on the total magnetic field B_{tot} (Haug et al. 1987, Chakraborty and Pietiläinen 1989, Halonen et al. 1990). They found that with increasing tilt angle, dramatic changes occur in the ρ_{xx} minima of various filling factors. In fig. 20 we present some of the experimental results of ρ_{xx} versus θ by Clark et al. Clearly, the $\frac{4}{3}$ state is first destroyed, followed by a reemergence as θ and hence the magnetic field is increased. The same effect was also observed for $\nu = \frac{2}{3}$. In contrast, the ρ_{xx} minima for $\frac{5}{3}$ and $\frac{1}{3}$ remain essentially unaltered with increasing tilt angle.

An important clue to understand these experimental results is the fact that, allowing for the spin degrees of freedom, the electron–hole symmetry (Zhang and Chakraborty

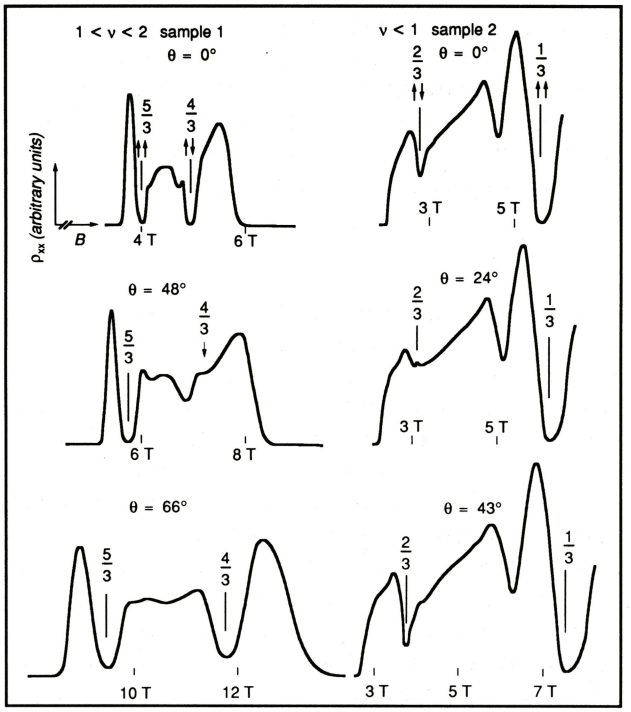

Fig. 20. Diagonal resistivity ρ_{xx} for various filling factors versus the tilted magnetic field.

1986) is not between ν and $1-\nu$, as described earlier, but between ν and $2-\nu$. Therefore, the $\frac{1}{3}$ and $\frac{5}{3}$ filling factors which are the spin-polarized states even at low magnetic fields, as predicted theoretically, remain unaffected by the tilted field. For the $\frac{2}{3}$ and $\frac{4}{3}$ states, the increasing magnetic field destroys the reversed-spin states (Maksym 1989) and, eventually, they reemerge as fully spin-polarized states.

Recently, the Cambridge group (Davies et al. 1991) reported very interesting FQHE experiments in high-quality p-type heterojunctions with tilted magnetic field. They found the same magnetic-field-dependent behavior of $\nu=\frac{4}{3}$ as observed in n-type heterojunctions by Clark et al. (1989). They also noted that for the two-dimensional hole system one requires a smaller magnetic field to destroy and return the $\frac{4}{3}$ state which suggests that the Zeeman splitting is larger for that system.

The other important tilted-field experiment indicative of spin-reversed states was by Eisenstein et al. (1989) who discovered a transition between two distinct FQHE states at the same filling factor $\nu=\frac{8}{5}$ (i.e., $2-\frac{2}{5}$). The transition is driven by tilting the magnetic field and as discussed below, the data quite consistently indicate a change from a spin-unpolarized fluid to a polarized fluid.

Eisenstein et al. then measured the activation energy W versus the tilt angle for $\nu=\frac{8}{5}$. Figure 21 depicts the angular dependence of the observed activation energy, which is the most interesting result as far as the spin-reversed quasiparticles are concerned. The results presented in fig. 21 are all obtained at a fixed filling factor

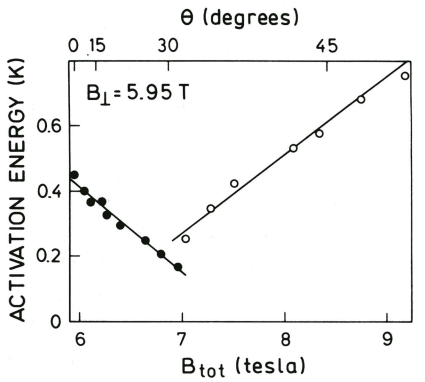

Fig. 21. Activation energy of the FQHE state at $\nu = \frac{8}{5}$ versus B_{tot}. Solid and open symbols refer to low-field and high-field components, respectively.

$\nu = \frac{8}{5}$ and hence a fixed perpendicular magnetic field $B_\perp \sim 5.95$ T. They are plotted against *total* magnetic field, $B_{tot} = B_\perp/\cos\theta$. As θ increases from zero, W *decreases* linearly. Beyond about 30°, W begins to rise again, eventually exceeding its value at $\theta = 0°$. For small angle (i.e., low magnetic field) the ground state is expected to be analogous to the two-spin $\frac{2}{5}$ state (see § 3.3). From the slope dW/dB_{tot} at small angles in fig. 21, it was found that the g-factor is $g \sim 0.4$ in remarkable agreement with recent spin-resonance measurements on two-dimensional electrons in GaAs (Dobers et al. 1988).

At low magnetic fields, the ground state of $\nu = \frac{8}{5}$ is expected to be spin-unpolarized and presumably the low-lying excitations involve spin-reversed quasiparticles. This could explain the linear decrease of the activation energy as the tilt angle (or the magnetic field) is increased. As the Zeeman energy at $\nu = \frac{8}{5}$ is increased by tilting the magnetic field, the polarized state is energetically favored and eventually becomes the new ground state. The low-lying excitations might still involve spin-reversed quasiparticles. That could explain the linear increase of the activation energy (Chakraborty et al. 1986, Chakraborty and Pietiläinen 1986) discussed in the preceding section. Further increase of the magnetic field would lead to a fully spin-polarized ground state with Laughlin-type quasiparticle–quasihole excitations and we would expect the usual $B^{1/2}$ behavior of the activation energy (Coulomb gap) seen in fig. 18. There has been much theoretical and experimental work recently on the spin-reversed ground state and excitations at various filling fractions (Buckthought et al. 1991, Chakraborty and Pietiläinen 1990, Chakraborty 1990, Maksym 1989, Clark et al. 1990, Eisenstein et al. 1990, Furneaux et al. 1989, Sachrajda et al. 1990, Engel et al. 1992).

4.2.6. Quantization condition

The Kubo formula approach for the quantization condition of the IQHE has been discussed in § 3.2.3. There we discussed the topological approach by Niu et al. (1985). In an important paper, Tao and Haldane (1986) presented arguments on the integral

and fractional quantization which are quite general and related to the above approaches. The arguments are, however, rather technical and cannot be reproduced here in their entirety. Therefore, we will mention briefly the essential results and refer the reader to the original article. For the electron system in a periodic rectangular geometry, Tao and Haldane (1986) and Haldane (1985) showed that the many-body Hamiltonian can be separated into two terms due to the center of mass motion and the internal relative motion. The center of mass moves as a charged particle in a periodic potential. If there is an energy gap at a filling factor, it is due to the relative motion. The Hall conductance is associated with the quantum motion of the center of mass and, therefore, the analysis of the quantization condition can be related to the topological approach discussed in § 3.2.3. Tao and Haldane (1986) also noticed that, in the presence of a weak concentration of impurities, the quantization condition remains unchanged. The Hall conductivity is still expressed in the topologically invariant form.

4.3. Other developments

4.3.1. Collective modes

In addition to the elementary charged excitations discussed in § 4.2.1, the FQHE system also supports collective excitations whose dispersion has been studied by various theoretical methods. In a spherical geometry, the energy-level spectrum for a finite-electron system at $\nu = \frac{1}{3}$ in the lowest Landau level was calculated by Haldane and Rezayi (1985) and Fano et al. (1986). In the case of a periodic rectangular geometry, the major contribution to the collective excitation spectrum was also calculated by Haldane (1985). He found that, at rational values of ν, the states could be characterized by a two-dimensional wave vector $\boldsymbol{k}$, and, hence, the collective excitation spectrum could be calculated in the periodic rectangular geometry.

The finite-size studies described in § 4.1.1 were carried out in an infinite lattice with a periodic rectangular geometry. The electrons in one cell of the lattice have identical mirror images in all other cells. This infinite repetition will introduce a symmetry which can be employed to classify the eigenstates of the Hamiltonian. This classification has the further advantage that the size of the matrix to be numerically diagonalized is reduced.

In the absence of a magnetic field, the symmetry analysis would be simple: we would have a translational group in the periodic lattice and the eigenstates could be labeled by the wave vectors in the inverse lattice. The physical interpretation of these quantum numbers would, of course, be the momentum. The presence of the magnetic field, however, slightly complicates the classification scheme. We could apply the well-known apparatus of group theory which would lead us to the so-called *ray representation* of the magnetic translation group (Brown 1968, Maksym 1985). The reason why we do not obtain an ordinary representation is that the symmetry operations of the group obey a noncommutative algebra where the product of operators is an operator of the same group only to within a phase factor. For every lattice vector $\boldsymbol{L}_{mn}$, there is a relative translation operator which commutes with the

Hamiltonian. The eigenvalues of this operator are $e^{2\pi i(ms+nt)/N}$, where N is the highest common divisor of N_e and N_s. The quantum numbers s and t ($s, t = 0, 1, \ldots, N-1$) are related to the physical momentum by

$$kl_0 = \left(\frac{2\pi}{N_s\lambda}\right)^{1/2}[s - s_0, \lambda(t - t_0)],$$

where the point (s_0, t_0), corresponding to the state $\boldsymbol{k} = 0$, is required to be the most symmetric point of the reciprocal lattice, and λ is the aspect ratio.

In fig. 22 we present the dispersion for the density wave mode obtained for four- to seven-electron systems in the lowest Landau level in a periodic rectangular geometry. Only the three lowest excitation energies are shown. The spectrum is, in fact, a function of the magnitude of the two-dimensional vector $\boldsymbol{k}$. The ground state is obtained at $\boldsymbol{k} = 0$, as expected. The lowest energy excitations are separated from the ground state by a large gap, which reflects the incompressible nature of the system and they clearly show a collective behavior with a minimum at finite kl_0. For small kl_0, the modes are not very well defined, as they are close to the continuum of the higher energy states. The numerical calculations were done by Yoshioka (1986a), but qualitatively similar results were also obtained earlier by Haldane (1985) for a six-electron system in a square geometry. Yoshioka (1986b) and Chakraborty and Pietiläinen (1988a) also calculated the excitation spectrum for spin-reversed systems.

The finite-size calculations discussed above provide quite accurate information for the collective excitations and the energy gap in the FQHE. However, not much physical insight is gained from these numerical calculations. The theoretical work by Girvin et al. (1985, 1986), drawing analogies from Feynman's well-known theory of liquid ^{4}He, is very helpful in this respect.

Given the exact ground-state ψ, the density-wave excited state at wave vector $\boldsymbol{k}$ is written as (Feynman 1972)

$$\varphi_{\boldsymbol{k}} = N^{-1/2}\rho_{\boldsymbol{k}}\psi, \tag{72}$$

with the density operator $\rho_{\boldsymbol{k}} \equiv \Sigma_{j=1}^{N} e^{-i\boldsymbol{k}\cdot\boldsymbol{r}_j}$, where N is the number of particles. Let

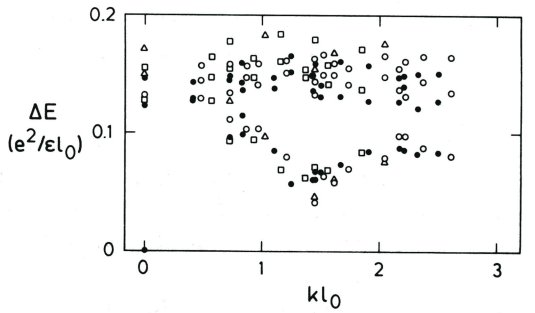

Fig. 22. Low-lying excitation energies for four- (△), five- (□), six- (○) and seven-electron (●) systems at $\nu = \frac{1}{3}$.

us define the oscillator strength as $f(k) \equiv N^{-1}\langle\psi|\rho_k^\dagger(\mathscr{H} - E_0)\rho_k|\psi\rangle$, where $\mathscr{H}$ is the Hamiltonian and E_0 is the ground-state energy. Let us also define the static structure function $s(k) \equiv N^{-1}\langle\psi|\rho_k^\dagger\rho_k|\psi\rangle$. The excitation energy is then given by the well-known formula

$$\Delta(k) = \frac{f(k)}{s(k)} = \frac{\hbar^2 k^2}{2m^* s(k)}. \tag{73}$$

The above result can be interpreted as saying that the collective mode energy is the single-particle energy $\hbar^2 k^2/2m^*$ renormalized by the static structure function representing correlations among the particles.

As is well known, for liquid ^{4}He there are no low-lying single-particle excitations and the only low-lying excitations are long-wavelength density oscillations – the *phonons*. The excitation energy versus wave vector curve vanishes linearly, its slope corresponding to the velocity of sound. Near $k = 2$ Å^{-1}, the excitation energy shows a *roton minimum*, which arises due to the peak of the static structure function.

In the case of the FQHE, if we insist that the excited state is entirely within the lowest Landau level, the density-wave excited state becomes

$$\varphi_k = N^{-1/2}\bar{\rho}_k\psi, \tag{74}$$

where $\bar{\rho}_k$ is the *projection* of the density operator onto the subspace of the lowest Landau level,

$$\bar{\rho}_k = \sum_{j=1}^{N} \exp\left[-ik\frac{\partial}{\partial z_j}\right]\exp[-\tfrac{1}{2}ik^* z_j], \tag{75}$$

where all derivatives operate to the left (Girvin et al. 1986). Also, the projected potential energy is

$$\bar{V} = \tfrac{1}{2}\int \frac{d\boldsymbol{q}}{(2\pi)^2} v(q)(\bar{\rho}_{\boldsymbol{q}}^\dagger\bar{\rho}_{\boldsymbol{q}} - \rho e^{-q^2 l_0^2/2}), \tag{76}$$

where $v(q)$ is the interaction potential [$v(q) = 2\pi/q$ in the present case]. The projected oscillator strength is

$$\bar{f}(k) = N^{-1}\langle 0|\bar{\rho}_k^\dagger[\bar{\mathscr{H}}, \bar{\rho}_k]|0\rangle, \tag{77}$$

where $|0\rangle$ is the ground state. Since the kinetic energy is constant, one can replace the Hamiltonian by the potential energy. Let us also define the projected static structure factor $\bar{s}(k)$ as

$$\bar{s}(q) = N^{-1}\langle 0|\bar{\rho}_k^\dagger\bar{\rho}_k|0\rangle. \tag{78}$$

In the single-mode approximation (so named because of the assumption that the density-wave alone saturates the full projected oscillator strength sum), the excitation energy is $\Delta(k) = \bar{f}(k)/\bar{s}(k)$. For small k, $\bar{f}(k)$ vanishes like $|k|^4$ and then for a gap to exist, $\bar{s}(k)$ must vanish as $\sim|k|^4$.

Using the Laughlin wave function for the ground state, the structure factor and the function $\bar{f}(k)$ can be computed numerically from the above relations. The resulting excitation energy obtained by Girvin et al. (1985, 1986) is presented in fig. 23 for

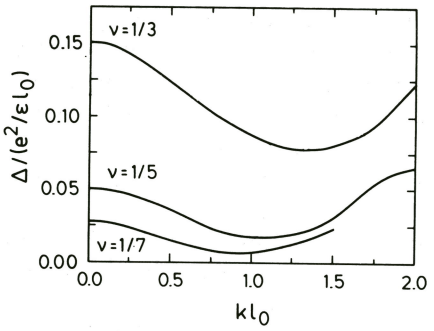

Fig. 23. Collective excitation curve in the single-mode approximation for $\nu = \frac{1}{3}, \frac{1}{5}$ and $\frac{1}{7}$ filling fractions.

$\nu = \frac{1}{3}, \frac{1}{5}$ and $\frac{1}{7}$. The low-lying excitation energy curve reveals several interesting features. The first thing to note is that, unlike ^{4}He, the collective mode has a finite gap at $k = 0$, i.e., the mode is *not* a massless Goldstone mode. It should be pointed out, however, that this gap is *not* due to the charged particles, since the Coulomb force is not sufficiently long-ranged to provide a finite plasma frequency in two dimensions (the plasma frequency goes to zero with a square root dependence on the wave vector). The finite gap originates from the incompressible nature of the electron system at some particular filling fractions. The collective mode also shows a minimum at a finite k. This minimum is due to the peak $\bar{s}(k)$ and is thus analogous to the *roton* minimum in liquid ^{4}He. For large wave vectors ($kl_0 \gg 1$) the density wave is no longer a suitable excitation and the single-mode approximation is no longer appropriate.

Unlike the elementary excitations discussed in § 4.2.1, no experimental results are available, as yet, for the collective mode described in this section. It should, however, be mentioned that roton-type structures expected in the inter-Landau-level excitations at integer filling factors (Kallin and Halperin 1984) have been observed by inelastic light-scattering experiments (Pinczuk et al. 1988, 1990). Similar excitations for a fractional filling of the lowest Landau level have also been studied theoretically (MacDonald et al. 1985, Pietiläinen and Chakraborty 1988, Pietiläinen 1988).

4.3.2. Even-denominator filling fractions

In our discussion of the theoretical work so far, we have only described the filling fractions with *odd* denominators. The fact that all investigations focused their attention mostly on these fractions is hardly surprising, because the experimental results clearly demonstrated that for the FQHE to occur, the odd denominators were apparently favored exclusively. As we recall, Laughlin's approach *explains* such a fact by the requirement of antisymmetry under interchange of particles and in the hierarchical scheme such filling factors are taken as the starting point in developing the higher-order filling factors with odd denominators. The possibility of observing the FQHE for *even* denominator filling factors is not excluded, however, in these theories.

The simplest filling factor with even denominator is $\nu = \frac{1}{2}$. In this case, the Laughlin-type wave function would describe a system of particles obeying *Bose* statistics. However, one can group the electrons into bound *pairs*, and the pairs can then transform as bosons under interchange of their positions (Halperin 1983) and a Laughlin-type wave function could still be used. For the finite-size system in a periodic rectangular geometry (§ 4.1.1), with particles obeying *Bose* statistics, a *cusp* at $\nu = \frac{1}{2}$ was, in fact, observed by Yoshioka (1984a). Finite-size studies in a spherical geometry were also performed by Fano et al. (1987). The pairing idea of Halperin has been recently revived in the theoretical work of Greiter et al. (1991, 1992) for this filling fraction.

The ground-state energy (per electron) at $\nu = \frac{1}{2}$, calculated for four- to ten-electron systems in a periodic rectangular geometry (Chakraborty and Pietiläinen 1988a) is shown in fig. 24. In contrast to the case of $\nu = \frac{1}{3}$, the results in this case show strong dependence on the electron number. The extrapolation of the results in the thermodynamic limit leads to the energy $\approx -0.472 e^2/\varepsilon l_0$. The results are, of course, lower compared to the Wigner crystal (WC) energy in the HF limit: the energy difference is ≈ 0.028, while for $\nu = \frac{1}{3}$, the corresponding energy difference is ≈ 0.025. However, the energy difference is much smaller (~ 0.01) for the crystal energies obtained for the four-electron system by Yoshioka et al. (1983) (see fig. 14). Given such a small difference, it is not possible to entirely rule out the crystal state at $\nu = \frac{1}{2}$. The Laughlin state energy at $\nu = \frac{1}{2}$, which corresponds to the boson system, is also given in fig. 24.

The excitation spectrum at this filling fraction bears no similarity to that for an incompressible fluid. Here the ground state appears at a finite $\boldsymbol{k}$ and varies strongly with the particle number and geometry of the cell. The whole spectrum is, in fact, particle-number- and geometry-dependent. No general conclusion can be drawn from those numerical results (Chakraborty and Pietiläinen 1988a).

Experimentally, the possibility of observing the FQHE at even denominator filling factors was indicated by several groups. A minimum in ρ_{xx} at $\nu = \frac{3}{4}$ was first observed by Ebert et al. (1984). Clark et al. (1986) observed minima in the diagonal resistivity

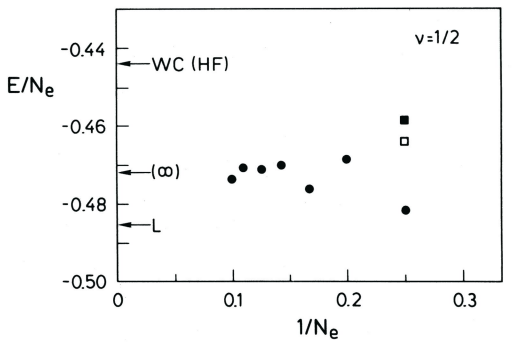

Fig. 24. Ground-state energy per electron at $\nu = \frac{1}{2}$ (in units of $e^2/\varepsilon l_0$) as a function of electron number in a periodic rectangular geometry. The closed and open squares correspond to the crystal energies of fig. 14, and the HF energy is given for comparison. The energy of the Laughlin state (depicted as L) and the extrapolation of the finite system results to the thermodynamic limit [depicted as (∞)] are also given.

in the second Landau level at $\nu = \frac{9}{4}, \frac{5}{2}$, and $\frac{11}{4}$. Correct quantization of ρ_{xy} to these fractional values was not achieved, however.

A thorough analysis of these filling factors has been performed by Willett et al. (1987). Their results for $\nu < 1$ (fig. 13) do not show any sign of the FQHE for even denominator fillings. While some features in ρ_{xx} were seen at $\nu = \frac{3}{4}$, two higher-order odd denominator fillings, $\nu = \frac{4}{5}$ and $\nu = \frac{5}{7}$, seem to converge toward this even denominator filling factor. For $\nu = \frac{1}{2}$, ρ_{xy} follows the classical straight line, while the broad minimum in ρ_{xx} is thought to be caused by, as yet, unresolved higher-order odd denominator filling factors.

For $3 > \nu > 2$ (the Landau level $n = 1$), however, the situation is entirely different. This region of filling factors in fig. 13 is presented in more detail in fig. 25. The ρ_{xy} curve shows a plateau at the magnetic field which corresponds to $\nu = \frac{5}{2}$, centered at $\rho_{xy} = (h/e^2)/\frac{5}{2}$ to within 0.5%. In the same region of magnetic field, a deep minimum is observed in ρ_{xx}.

There have been a few theoretical attempts to explain the experimental findings discussed above. Haldane and Rezayi (1988) proposed a spin-singlet wave function for an incompressible state which occurs at $\nu = \frac{1}{2}$. From small-system calculations (six electrons) with model pseudopotentials, they concluded that such a state might be responsible for the $\frac{5}{2}$ effect. Numerical calculations were performed for the Coulomb interaction by Chakraborty and Pietiläinen (1988b). The calculations were also based on finite-size systems (up to six electrons in a periodic rectangular geometry), and do not support the conclusions of Haldane and Rezayi. In table 2, we present the ground-state energies for four- and six-electron systems at $\nu = \frac{1}{2}$ in the second Landau level for different spin polarizations. The system has a fully spin-polarized ground state, even in the absence of the Zeeman energy. The energy difference between the various spin states is very small. However, for a spin-reversed system to be energetically favored, the energy of this state must be larger than that of the spin-

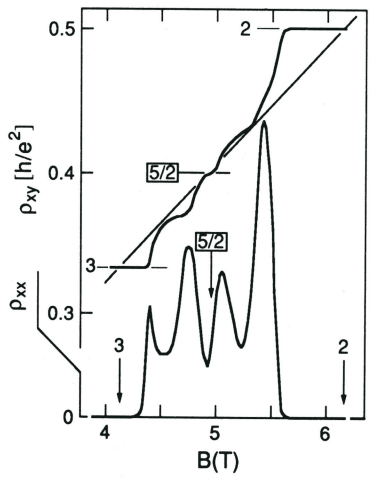

Fig. 25. The region of filling factors $3 > \nu > 2$ of fig. 13 shown in detail at $T = 25$ mK.

Table 2
Ground-state energies (in units of $e^2/\varepsilon l_0$) for four- and six-electron systems at $\nu = \frac{5}{2}$ for various values of the total spin S. The Zeeman energy is not included in the energy values.

N_e	$S=0$	$S=1$	$S=2$	$S=3$
4	−0.3644	−0.3655	−0.3849	–
6	−0.3782	−0.3783	−0.3785	−0.3797

polarized state by at least the Zeeman energy contribution. Such a situation might occur for larger systems than the ones considered here, as the even-denominator system results are known to be very system-size dependent.

In a tilted-field measurement on the $\frac{5}{2}$ state, Eisenstein et al. (1988) observed that the FQHE collapses rapidly at this filling factor. This indicates that the spin degree of freedom is perhaps playing a role. However, from the above discussions, it is fair to conclude that finite-size calculations performed so far are unable to provide a suitable explanation of the FQHE at this interesting filling factor.

Some recent experiments have unearthed several puzzling facts about the filling factors $\nu = \frac{1}{2}$ and $\frac{3}{2}$ whose origins are far from clear. Jiang et al. (1989) observed deep low-temperature minima in ρ_{xx} at these filling fractions. Although their strength exceeds the strength of neighboring FQHE states, there are no plateaus nor any discernible indication of plateau developments visible in ρ_{xy}. The states at half-filling are distinctly different from the FQHE states because of the unusual T-dependence in ρ_{xx} and the lack of plateaus in ρ_{xy}. Transport measurements in tilted magnetic fields seem not to alter the above features.

4.3.3. Multiple layer systems

In this section, we present the theoretical and experimental work on the FQHE in a *layered* electron system. Multilayer electron systems have been studied earlier quite extensively as an anisotropic model for an electron gas. Let us consider a model where two layers with equal density of electrons are embedded in an infinite dielectric. We consider the delta-function-localized electron density in each plane. The electrons move freely in each plane and the interaction of electrons in different planes is considered to be Coulombic. Tunneling of electrons between the two planes is not allowed. The electrons are also considered to be in their lowest subband. This model is often referred to in the literature as the Visscher–Falicov model (Visscher and Falicov 1971).

In fig. 26, we have presented the excitation spectrum for a two-layer system (layers separated by a distance C) with four electrons per layer in a periodic rectangular geometry (Chakraborty and Pietiläinen 1987, 1988c), with an aspect ratio $\lambda = 1.25$. The first important result is that the ground state is obtained uniquely at $\boldsymbol{k} = 0$. It remains so for different aspect ratios. The other interesting result is that a gap structure in the spectrum is obtained with a characteristic minimum at a finite kl_0, similar to that of the magnetoroton minimum, discussed in § 4.3.1.

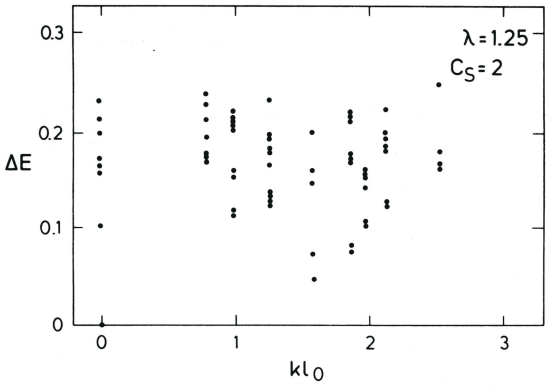

Fig. 26. Excitation energy (in units of $e^2/\varepsilon l_0$) of an eight-electron system in a two-layer geometry at $\nu = \frac{1}{2}$ for a dimensionless layer separation parameter $C_s = C/l_0 = 2.0$.

A comparison of the layered-system results with those of the single layer indicates that the introduction of an interacting electron layer has helped to reorganize the excitation energies of the system, particularly the $\boldsymbol{k} = 0$ state (Chakraborty and Pietiläinen 1988a). The observation of the roton-type minimum is also quite interesting. The results indicate the possibility of the occurrence of an incompressible fluid state at $\nu = \frac{1}{2}$ in a multiple layer system.

There have been a few other calculations on the two-layered systems. Yoshioka et al. (1989) obtained the ground-state wave functions for finite-size systems on a sphere and compared them with the Jastrow-type wave functions at filling factors $\nu = 1, \frac{1}{2}, \frac{2}{5}$, and $\frac{1}{3}$. Halperin's two-spin state wave function, eq. (70), was employed in that work. Fertig (1989) investigated the excitation spectrum of two- and three-layer systems with filling factors $\nu = \frac{1}{2}$ and $\nu = \frac{1}{3}$, respectively, in each layer. The excitation spectra were calculated following the scheme of Kallin and Halperin (1984) for magnetoplasmon work. He found that the excitation spectrum has soft modes when $C > 1.21 l_0$ for the two-layer system and $C_1 > 0.92 l_0$, $C_2 > 1.51 l_0$ for the three-layer system. This indicates that the system undergoes a phase transition as the layer spacing is increased through these critical separations.

Experimental attempts to explore the FQHE in two-layer systems have been reported recently. Lindelof et al. (1989) reported an interesting observation of $\nu = \frac{1}{2}$ in each of the two parallel layers of a selectively doped double heterostructure. Other experimental results with that sample indicate that the system indeed consists of two parallel and independent two-dimensional electron layers. Experiments performed by Eisenstein et al. (1992) also show results in agreement with the theoretical results discussed above.

4.3.4. Nature of long-range order in the Laughlin state

The analogy of the FQHE state with liquid ^{4}He has been quite apparent in the study of collective excitations discussed in § 4.3.1, in particular, the presence of magnetorotons in the excitation spectrum. It is, therefore, quite natural to look for

some other interesting properties in the FQHE state which are known to exist in ^{4}He, notable among which is the so-called off-diagonal long-range order (ODLRO) in the density matrices (Sears 1985).

In the case of the QHE, the off-diagonal elements of the one-body density matrix in real space exhibit Gaussian decay (MacDonald and Girvin 1988)

$$\rho_1(\boldsymbol{r}, \boldsymbol{r}') = \frac{\nu}{2\pi} \mathrm{e}^{-|z-z'|^2/4} \mathrm{e}^{(z^* z' - z z'^*)/4}. \tag{79}$$

Therefore, no ODLRO exists in the one-body density matrix of the two-dimensional electron system.

To exhibit ODLRO, a fermionic system has to form at least pairs. This is the case in the BCS theory of traditional superconductivity where the two-body density matrix has extensive eigenvalues (Yang 1962). Some years ago, Thouless (1985) investigated, in the context of the FQHE, the corresponding density matrix

$$\rho_{m_1, m_2}(n) = \langle a^\dagger_{n+m_1} a^\dagger_{n-m_1} a_{n-m_2} a_{n+m_2} \rangle, \tag{80}$$

where, for fixed n, the eigenvalues were determined for finite-electron systems in a spherical geometry, as well as for the Laughlin wave function on a square with periodic boundary condition in one direction. The existence of Cooper pairs with *wave number* $2n$ is expected to be indicated by a large eigenvalue of $\rho(n)$. The largest eigenvalues in those calculations, however, were found to *decrease* rather than increase with the number of electrons. The results, therefore, indicate that there is no sign of any order in the two-body density matrix for the FQHE.

Although electrons in a strong magnetic field do not exhibit ODLRO, it seems likely that there might be some order hidden in the space of those quantum states. In the pursuit of such an order a singular gauge field $\mathscr{A}_j$ used in the study of *anyons* (see § 4.2.2), has been considered $\mathscr{A}_j(z_j) = (\lambda \boldsymbol{\Phi}_0 / 2\pi) \Sigma_{i \neq j} \nabla_j \Im \ln(z_j - z_i)$ (Girvin and MacDonald 1987), where $\boldsymbol{\Phi}_0$ is the flux quantum and λ is a constant. It corresponds to a vector potential that would be included in the Hamiltonian if each particle had attached to itself a solenoid carrying $\frac{1}{2}\lambda$ flux quanta. It should be noted that adding this vector potential to the Hamiltonian we do not make a true gauge transformation since a flux tube is attached to each particle. However, if $\lambda = m$, an integer, the net effect is to change the phase of the wave function,

$$\psi_{\mathrm{new}} = \exp\left[-im \sum_{i<j} \Im \ln(z_i - z_j) \right] \psi_{\mathrm{old}}. \tag{81}$$

If ψ_{old} is the Laughlin wave function (45) we get the transformed state

$$\tilde{\psi}(z_1, \ldots, z_N) = \prod_{i<j} |z_i - z_j|^m \exp\left(-\tfrac{1}{4} \sum_k |z_k|^2 \right), \tag{82}$$

which is purely real and is symmetric under particle exchange for both even and odd m. This is truly a remarkable result that both fermion and boson systems map onto bosons in this singular gauge.

Making use of the plasma analogy discussed in § 4.1, one obtains

$$\tilde{\rho}(z, z') = \left(\frac{\nu}{2\pi}\right) \exp[-\beta\, \Delta f(z, z')]\, |z - z'|^{-m/2}, \tag{83}$$

where $\beta \equiv 2/m$ and $\Delta f(z, z')$ is the difference in free energy between the two impurities of charge $\frac{1}{2}m$ (located at z and z') and a single impurity of charge m (with arbitrary location). The asymptotic value of Δf can be computed using thermodynamic integration of the screening charge density. For general values of m the screening charge distribution can be found using the ion-disk approximation or linear response based on the known static structure factor of the plasma (Caillol et al. 1982). As the plasma screens the impurities completely, the free energy difference $\Delta f(z, z')$ rapidly approaches a constant (fig. 27) as the separation $|z - z'| \to \infty$. The one-body density matrix, therefore, shows a *power-law* decay in the long-range. This implies that there is no ODLRO even in the transformed Laughlin state. It should be remembered that in the *ground state* the density matrix should attain a finite value for infinite separation if the ODLRO exists in that system *even in two dimensions* (Reatto and Chester 1967).

A physical interpretation of the power-law decay of the one-body density matrix for the transformed Laughlin state has been provided by Chakraborty and von der Linden (1990). In determining the one-body density matrix for the modified Laughlin wave function for N electrons, one can map the wave function onto a classical plasma with $N - 1$ plasma particles and two additional *phantom* particles residing at sites z and z'. These particles have charges half as large as the original plasma particles and experience an interaction with the plasma particles but not with each other. The one-body density matrix can then be related to the difference in free energy $\Delta F(z, z')$ between this system and the N-particle classical plasma, with inverse temperature $\beta = 1/m$: $\rho(z, z') = \rho_0\, e^{\beta \Delta F(z, z')}$, and $\rho_0 = 1/2\pi m$ being the plasma density. In a two-component system, the one-body density matrix is, in fact, related to the pair-correlation function of the phantom particles in the zero-concentration limit,

$$\rho(z, z') = \lim_{\rho_\gamma \to 0} \{[g_{\gamma\gamma}(0, 0)]^{-1}\, g_{\gamma\gamma}(z, z')\}, \tag{84}$$

where γ stands for the phantom particles. For systems like ^{4}He, with interactions

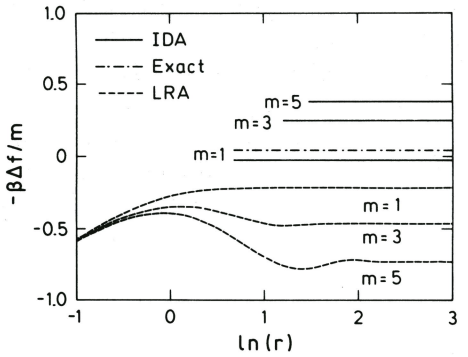

Fig. 27. Plot of $-\beta\, \Delta f(z, z')/m$ versus $r \equiv |z - z'|$ for filling factor $\nu = 1/m$. LRA is linear response approximation. IDA is ion-disk approximation.

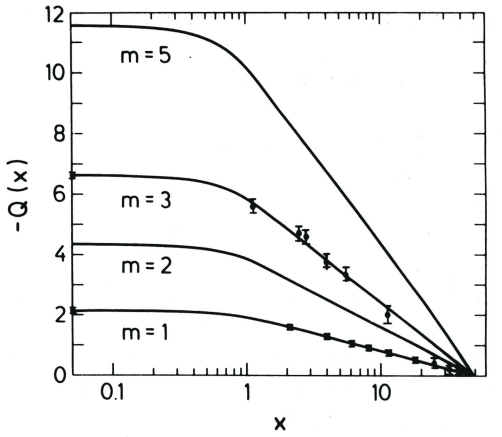

Fig. 28. Plot of $-Q(x)$ versus the interparticle separation x, measured in units of the mean electronic separation $r_0 = \sqrt{2m}\, l_0$. The HNC results (solid lines) are compared with the Monte Carlo data (solid circles).

vanishing at large distances, the asymptotic value of $g(z, z')$ is unity and, therefore, the asymptotic value of $\rho(z, z')$ is given by the inverse of the pair-correlation function at the origin. This is not true for systems with *increasing* interactions like the classical plasma in two dimensions.

Chakraborty and von der Linden (1990) performed both HNC calculations and Monte Carlo calculations for up to 1000 particles, in the case of logarithmically increasing interactions. The numerical results of the HNC and Monte Carlo studies are plotted in fig. 28, the conventional definition of the density matrix is used in the HNC scheme: $\rho(z, z') = n_c\, e^{-Q(|z-z'|)}$, where $n_c = g_{\gamma\gamma}(0, 0)^{-1}$ is the conventional condensate fraction and $Q(z, z')$ is related to the pair-correlation functions via the HNC equations (Puoskari and Kallio 1984, Lam and Ristig 1979, Fantoni 1978). The interparticle separation in these calculations is large enough that the asymptotic behavior of $\tilde{\rho}$ for various m can be extracted very accurately. From this plot we readily obtain the power $-\frac{1}{2}m$ in agreement with eq. (83). The values for $-\beta\Delta f_\infty/m$ are 0.02 for $m = 1$ and 0.24 for $m = 3$ and are in fairly good agreement with the earlier results (Girvin and MacDonald 1987). The screening length is of order $\sqrt{2m}\, l_0$. The values of n_c obtained in the HNC scheme are 0.121 for $m = 1$, 0.014 for $m = 2$, 0.0015 for $m = 3$, and ~ 0 for $m = 5$. The comparison of the density matrix results via HNC with the corresponding Monte Carlo data show once more that HNC is a highly accurate approximation scheme, even in the case of increasing interactions.

Acknowledgements

I wish to acknowledge long and fruitful collaboration with Dr. Pekka Pietiläinen, who co-authored most of our papers cited. Over the years, I have benefitted from

discussions with numerous people, in particular, Bob Clark, Jim Eisenstein, Bert Halperin, Klaus von Klitzing, Allan MacDonald, Aron Pinczuk, and Horst Störmer. Excellent support from the colleagues at the Max-Planck-Institute, Stuttgart is gratefully acknowledged. I would like to thank Geof Aers for critically reading the manuscript and Marie D'Iorio and V.M. Pudalov for helpful discussions.

References

Abrahams, E., P.W. Anderson, D.C. Licciardello and T.V. Ramakrishnan, 1979, Phys. Rev. Lett. **42**, 673.
Aers, G.C., and A.H. MacDonald, 1984, J. Phys. C **17**, 5491.
Ando, T., 1982, Anderson Localization, eds Y. Nagaoka and H. Fukuyama (Springer, Berlin) p. 176.
Ando, T., 1983a, J. Phys. Soc. Jpn. **52**, 1740.
Ando, T., 1983b, Recent Topics in Semiconductor Physics, eds H. Kamimura and Y. Toyozawa (World Scientific, Singapore) p. 72.
Ando, T., 1984, J. Phys. Soc. Jpn. **53**, 3101, 3126.
Ando, T., 1986, Surf. Sci. **170**, 243.
Ando, T., and H. Aoki, 1985, J. Phys. Soc. Jpn. **54**, 2238.
Ando, T., and Y. Uemura, 1974, J. Phys. Soc. Jpn. **39**, 959.
Ando, T., Y. Matsumoto and Y. Uemura, 1975, J. Phys. Soc. Jpn. **39**, 279.
Ando, T., A.B. Fowler and F. Stern, 1982, Rev. Mod. Phys. **54**, 437.
Aoki, H., 1982, J. Phys. C **15**, L1227.
Aoki, H., 1987, Rep. Prog. Phys. **50**, 655.
Aoki, H., and T. Ando, 1981, Solid State Commun. **38**, 1079.
Aoki, H., and T. Ando, 1985, Phys. Rev. Lett. **54**, 831.
Aoki, H., and T. Ando, 1986a, Phys. Rev. Lett. **57**, 3093.
Aoki, H., and T. Ando, 1986b, Surf. Sci. **170**, 249.
Apal'kov, V.M., and E.I. Rashba, 1991, JETP Lett. **56**, 49.
Apenko, S.M., and Yu.E. Lozovik, 1985, Sov. Phys.-JETP **62**, 328.
Arovas, D., J.R. Schrieffer and F. Wilczek, 1984, Phys. Rev. Lett. **53**, 722.
Avron, J.E., and R. Seiler, 1985, Phys. Rev. Lett. **54**, 259.
Azbel, M.Ya., 1985, Solid State Commun. **53**, 147.
Baraff, G.A., and D.C. Tsui, 1981, Phys. Rev. B **24**, 2274.
Beenakker, C.W.J., and H. van Houten, 1991, Solid State Phys. **44**, 1.
Boebinger, G.S., A.M. Chang, H.L. Störmer and D.C. Tsui, 1985, Phys. Rev. Lett. **55**, 1606.
Boebinger, G.S., H.L. Störmer, D.C. Tsui, A.M. Chang, J.C.M. Hwang, A.Y. Cho, C.W. Tu and G. Weimann, 1987, Phys. Rev. B **36**, 7919.
Brézin, E., D.J. Gross and C. Itzykson, 1984, Nucl. Phys. B **235** [FS 11], 24.
Brown, E., 1968, Solid State Phys. **22**, 313.
Buckthought, A., R. Boulet, A. Sachrajda, Z. Wasilewski, P. Zawadzki and F. Guillon, 1991, Solid State Commun. **78**, 191.
Buhmann, H., W. Joss, K. von Klitzing, I.V. Kukushkin, G. Martinez, A.S. Plaut, K. Ploog and V.B. Timofeev, 1990, Phys. Rev. Lett. **65**, 1056.
Buhmann, H., W. Joss, K. von Klitzing, I.V. Kukushkin, A.S. Plaut, G. Martinez, K. Ploog and V.B. Timofeev, 1991, Phys. Rev. Lett. **66**, 926.
Büttiker, M., 1988, Phys. Rev. B **38**, 9375.
Bychkov, Yu.A., and E.I. Rashba, 1989, Sov. Phys.-JETP **69**, 430.
Bychkov, Yu.A., S.V. Iordanskii and G.M. Eliashberg, 1981, JETP Lett. **33**, 143.
Byers, N., and C.N. Yang, 1961, Phys. Rev. Lett. **7**, 46.
Cage, M.E., R.F. Dziuba, B.F. Field, E.R. Williams, S.M. Girvin, A.C. Gossard, D.C. Tsui and R.J. Wagner, 1983, Phys. Rev. Lett. **51**, 1374.
Caillol, J.M., D. Levesque, J.J. Weis and J.P. Hansen, 1982, J. Stat. Phys. **28**, 325.

Chakraborty, T., 1985, Phys. Rev. B **31**, 4026.
Chakraborty, T., 1986, Phys. Rev. B **34**, 2926.
Chakraborty, T., 1990, Surf. Sci. **229**, 16.
Chakraborty, T., and P. Pietiläinen, 1986, Phys. Scr. **T14**, 58.
Chakraborty, T., and P. Pietiläinen, 1987, Phys. Rev. Lett. **59**, 2784.
Chakraborty, T., and P. Pietiläinen, 1988a, The Fractional Quantum Hall Effect (Springer, New York, Berlin, Heidelberg).
Chakraborty, T., and P. Pietiläinen, 1988b, Phys. Rev. B **38**, 10 097.
Chakraborty, T., and P. Pietiläinen, 1988c, Recent Progress in Many-Body Theories, eds A. Kallio, E. Pajane and R.F. Bishop (Plenum, New York) p. 113.
Chakraborty, T., and P. Pietiläinen, 1989, Phys. Rev. B **39**, 7971.
Chakraborty, T., and P. Pietiläinen, 1990, Phys. Rev. B **41**, 10 862.
Chakraborty, T., and P. Pietiläinen, 1991, Phys. Rev. B **44**, 13 078.
Chakraborty, T., and W. von der Linden, 1990, Phys. Rev. B **41**, 7872.
Chakraborty, T., and F.C. Zhang, 1984, Phys. Rev. B **29**, 7032.
Chakraborty, T., P. Pietiläinen and F.C. Zhang, 1986, Phys. Rev. Lett. **57**, 130.
Chalker, J.T., and P.D. Coddington, 1988, J. Phys. C **21**, 2665.
Chang, A.M., P. Berglund, D.C. Tsui, H.L. Störmer and J.C.M. Hwang, 1984, Phys. Rev. Lett. **53**, 997.
Clark, R., and P. Maksym, 1989, Physics World **2**, 39.
Clark, R.G., R.J. Nicholas, A. Ulsher, C.T. Foxon and J.J. Harris, 1986, Surf. Sci. **170**, 141.
Clark, R.G., J.R. Mallett, S.R. Haynes, J.J. Harris and C.T. Foxon, 1988, Phys. Rev. Lett. **60**, 1747.
Clark, R.G., S.R. Haynes, A.M. Suckling, J.R. Mallett, P.A. Wright, J.J. Harris and C.T. Foxon, 1989, Phys. Rev. Lett. **62**, 1536.
Clark, R.G., S.R. Haynes, J.V. Branch, A.M. Suckling, P.A. Wright, P.M.W. Oswald, J.J. Harris and C.T. Foxon, 1990, Surf. Sci. **229**, 25.
Davies, A.G., R. Newbury, M. Pepper, J.E.F. Frost, D.A. Ritchie and G.A.C. Jones, 1991, Phys. Rev. B **44**, 13 128.
D'Iorio, M., V.M. Pudalov and S.G. Semenchinsky, 1990, Phys. Lett. **150**, 422.
D'Iorio, M., J.W. Campbell, V.M. Pudalov and S.G. Semenchinsky, 1992, Surf. Sci. **263**, 49.
Dobers, M., K. von Klitzing and G. Weimann, 1988, Phys. Rev. B **38**, 5453.
Ebert, G., K. von Klitzing, C. Probst and K. Ploog, 1982, Solid State Commun. **44**, 95.
Ebert, G., K. von Klitzing, K. Ploog and G. Weimann, 1983, J. Phys. C **16**, 5441.
Ebert, G., K. von Klitzing, J.C. Maan, G. Remenyi, C. Probst, G. Weimann and W. Schlapp, 1984, J. Phys. C **17**, L775.
Eisenstein, J.P., H.L. Störmer, V. Narayanamurti, A.Y. Cho, A.C. Gossard and C.W. Tu, 1985, Phys. Rev. Lett. **55**, 875.
Eisenstein, J.P., R. Willet, H.L. Störmer, D.C. Tsui, A.C. Gossard and J.H. English, 1988, Phys. Rev. Lett. **61**, 997.
Eisenstein, J.P., H.L. Störmer, L. Pfeiffer and K.W. West, 1989, Phys. Rev. Lett. **62**, 1540.
Eisenstein, J.P., H.L. Störmer, L.N. Pfeiffer and K.W. West, 1990, Phys. Rev. B **41**, 7910.
Eisenstein, J.P., G.S. Boebinger, L. Pfeiffer, K. West and Song He, 1992, Phys. Rev. Lett. **68**, 1383.
Engel, L.W., S.W. Hwang, T. Sajoto, D.C. Tsui and M. Shayegan, 1992, Phys. Rev. B **45**, 3418.
Englert, T., and K. von Klitzing, 1978, Surf. Sci. **73**, 70.
Fano, G., F. Ortolani and E. Colombo, 1986, Phys. Rev. B **34**, 2670.
Fano, G., F. Ortolani and E. Tosatti, 1987, Nuovo Cimento **9**, 1337.
Fantoni, S., 1978, Nuovo Cimento A**44**, 191.
Fertig, H., 1989, Phys. Rev. B **40**, 1087.
Feynman, R.P., 1972, Statistical Physics (Benjamin, Reading, MA) ch. 11.
Fock, V., 1928, Z. Phys. **47**, 446.
Fukuyama, H., P.M. Platzman and P.W. Anderson, 1979, Phys. Rev. B **19**, 5211.
Furneaux, J.E., D.A. Syphers and A.G. Swanson, 1989, Phys. Rev. Lett. **63**, 1098.
Girvin, S.M., and A.H. MacDonald, 1987, Phys. Rev. Lett. **58**, 1252.
Girvin, S.M., A.H. MacDonald and P.M. Platzman, 1985, Phys. Rev. Lett. **54**, 581.
Girvin, S.M., A.H. MacDonald and P.M. Platzman, 1986, Phys. Rev. B **33**, 2481.

Giuliani, G.F., J.J. Quinn and S.C. Ying, 1983, Phys. Rev. B **28**, 2969.
Goldberg, B.B., D. Heiman, A. Pinczuk, L. Pfeiffer and K. West, 1990, Phys. Rev. Lett. **65**, 641.
Goldman, V.J., M. Shayegan and D.C. Tsui, 1988, Phys. Rev. Lett. **61**, 881.
Gorkov, L.P., A.I. Larkin and D.E. Khmel'nitzkii, 1979, JETP Lett. **30**, 228.
Gornik, E., 1987, The Physics of The Two-Dimensional Electron Gas, eds J.T. Devreese and F.M. Peeters (Plenum, New York) p. 365.
Gornik, E., R. Lassnig, G. Strasser, H.L. Störmer, A.C. Gossard and W. Wiegmann, 1985, Phys. Rev. Lett. **54**, 1820.
Greiter, M., X.G. Wen and F. Wilczek, 1991, Phys. Rev. Lett. **66**, 3205.
Greiter, M., X.G. Wen and F. Wilczek, 1992, Nucl. Phys. B **374**, 567.
Grimes, C.G., 1978, Surf. Sci. **73**, 379.
Guldner, Y., J.P. Hirtz, J.P. Vieren, P. Voisin, M. Voos and M. Razeghi, 1982, J. Phys. Lett. **43**, L613.
Guldner, Y., J.P. Vieren, M. Voos, F. Delahaye, D. Dominguez, J.P. Hirtz and M. Razeghi, 1986, Phys. Rev. B **33**, 3990.
Gusev, G.M., Z.D. Kvon, I.G. Neizvestnyi, V.N. Ovsyuk and P.A. Cheremnykh, 1984, JETP Lett. **39**, 541.
Hajdu, J., and G. Landwehr, 1985, Strong and Ultrastrong Magnetic Fields and Their Applications, ed. F. Herlach (Springer, Berlin).
Haldane, F.D.M., 1983, Phys. Rev. Lett. **51**, 605.
Haldane, F.D.M., 1985, Phys. Rev. Lett. **55**, 2095.
Haldane, F.D.M., and E.H. Rezayi, 1985, Phys. Rev. Lett. **54**, 237.
Haldane, F.D.M., and E.H. Rezayi, 1988, Phys. Rev. Lett. **60**, 956.
Halonen, V., T. Chakraborty and P. Pietiläinen, 1990, Phys. Rev. B **41**, 10 202.
Halperin, B.I., 1982, Phys. Rev. B **25**, 2185.
Halperin, B.I., 1983, Helv. Phys. Acta **56**, 75.
Halperin, B.I., 1984, Phys. Rev. Lett. **52**, 1583, 2390(E).
Halperin, B.I., 1986, Surf. Sci. **170**, 115.
Hansen, J.P., and D. Levesque, 1981, J. Phys. C **14**, L603.
Haug, R.J., K. von Klitzing, R.J. Nicholas, J.C. Maan and G. Weimann, 1987, Phys. Rev. B **36**, 4528.
Haug, R.J., J. Kucera, P. Streda and K. von Klitzing, 1989, Phys. Rev. B **39**, 10892.
Hikami, S., 1984, Phys. Rev. B **29**, 3726.
Hikami, S., 1986, Prog. Theor. Phys. **76**, 1210.
Huckestein, B., 1990, Physica A**167**, 175.
Huckestein, B., and B. Kramer, 1990, Phys. Rev. Lett. **64**, 1437.
Huo, Y., and R.N. Bhatt, 1992, Phys. Rev. Lett. **68**, 1375.
Ioffe, L.B., and A.I. Larkin, 1981, Sov. Phys.-JETP **54**, 556.
Iordansky, S.V., 1982, Solid State Commun. **43**, 1.
Jiang, H.W., H.L. Störmer, D.C. Tsui, L.N. Pfeiffer and K.W. West, 1989, Phys. Rev. B **40**, 12 013.
Jiang, H.W., R.L. Willett, H.L. Störmer, D.C. Tsui, L.N. Pfeiffer and K.W. West, 1990, Phys. Rev. Lett. **65**, 633.
Joynt, R., and R.E. Prange, 1984, Phys. Rev. B **29**, 3303.
Kallin, C., and B.I. Halperin, 1984, Phys. Rev. B **30**, 5655.
Kazarinov, R.F., and S. Luryi, 1982, Phys. Rev. B **25**, 7626.
Khmel'nitzkii, D.E., 1983, JETP Lett. **38**, 552.
Khmel'nitzkii, D.E., 1984, Phys. Lett. A**106**, 182.
Kirk, W.P., P.S. Kobiela, R.A. Schiebel and M.A. Reed, 1986, J. Vac. Sci. & Technol. A **4**, 2132.
Komiyama, S., T. Takamasu, S. Hiyamizu and S. Sasa, 1985, Solid State Commun. **54**, 479.
Kravchenko, S.V., V.M. Pudalov, J. Campbell and M. D'Iorio, 1992, JETP Lett. **54**, 532.
Kubo, R., 1957, J. Phys. Soc. Jpn. **12**, 570.
Kubo, R., S.J. Miyake and N. Hashitsume, 1965, Solid State Phys. **17**, 269.
Lam, P.K., and S.M. Girvin, 1984, Phys. Rev. B **30**, 473.
Lam, P.M., and M.L. Ristig, 1979, Phys. Rev. B **20**, 1960.
Landau, L., 1930, Z. Phys. **64**, 629.
Landauer, R., 1970, Philos. Mag. **21**, 863.

Laughlin, R.B., 1981, Phys. Rev. B **23**, 5632.
Laughlin, R.B., 1983a, Phys. Rev. Lett. **50**, 1395.
Laughlin, R.B., 1983b, Phys. Rev. B **27**, 3383.
Laughlin, R.B., 1984a, Springer Series in Solid State Sciences, Vol. 53, eds G. Bauer, F. Kuchar and H. Heinrich (Springer, Heidelberg) p. 272.
Laughlin, R.B., 1984b, Surf. Sci. **142**, 163.
Laughlin, R.B., 1987, The Quantum Hall Effect, eds R.E. Prange and S.M. Girvin (Springer, Berlin) p. 233.
Laughlin, R.B., M.L. Cohen, M. Kosterlitz, H. Levine, S.B. Libby and A.M.M. Pruisken, 1985, Phys. Rev. B **32**, 1311.
Leinaas, J.M., and J. Myrheim, 1977, Nuovo Cimento **37**, 1.
Levesque, D., J.J. Weis and A.H. MacDonald, 1984, Phys. Rev. B **30**, 1056.
Levine, H., S.B. Libby and A.M.M. Pruisken, 1983, Phys. Rev. Lett. **51**, 1915.
Licciardello, D.C., and D.J. Thouless, 1978, J. Phys. C **11**, 925.
Lindelof, P.E., H. Bruus, R. Taboryski and C.B. Sørensen, 1989, Semicond. Sci. & Technol. **4**, 858.
Luryi, S., and R.F. Kazarinov, 1983, Phys. Rev. B **27**, 1386.
MacDonald, A.H., 1989, The Quantum Hall Effect: A Perspective (Jaca Books, Milano).
MacDonald, A.H., and G.C. Aers, 1984, Phys. Rev. B **29**, 5976.
MacDonald, A.H., and S.M. Girvin, 1988, Phys. Rev. B **38**, 6295.
MacDonald, A.H., and P. Streda, 1984, Phys. Rev. B **29**, 1616.
MacDonald, A.H., H.C.A. Oji and S.M. Girvin, 1985, Phys. Rev. Lett. **55**, 2208.
Maksym, P.A., 1985, J. Phys. C **18**, L433.
Maksym, P.A., 1989, J. Phys.: Condens. Matter **1**, L6299.
Mallett, J.R., R.G. Clark, R.J. Nicholas, R. Willett, J.J. Harris and C.T. Foxon, 1988, Phys. Rev. B **38**, 2200.
Mendez, E.E., L. Esaki and L.L. Chang, 1985, Phys. Rev. Lett. **55**, 2216.
Mil'nikov, G.V., and I.M. Sokolov, 1988, JETP Lett. **48**, 536.
Monarkha, Y.P., and V. Shikin, 1982, Sov. J. Low Temp. Phys. **8**, 279.
Morandi, G., 1988, Quantum Hall Effect (Bibliopolis, Napoli).
Morf, R., and B.I. Halperin, 1986, Phys. Rev. B **33**, 2221.
Müller, G., D. Weiss, S. Koch, K. von Klitzing, H. Nickel, W. Schlapp and R. Lösch, 1990, Phys. Rev. B **42**, 7633.
Nicholas, R.J., M.A. Brummell, J.C. Portal, M. Razeghi and M. Poission, 1982, Solid State Commun. **43**, 825.
Niu, Q., D.J. Thouless and Y.S. Wu, 1985, Phys. Rev. B **31**, 3372.
Ono, Y., 1982, J. Phys. Soc. Jpn. **51**, 237.
Paalanen, M.A., D.C. Tsui and A.C. Gossard, 1982, Phys. Rev. B **25**, 5566.
Pietiläinen, P., 1988, Phys. Rev. B **38**, 4279.
Pietiläinen, P., and T. Chakraborty, 1988, Europhys. Lett. **5**, 157.
Pinczuk, A., J.P. Valladares, D. Heiman, A.C. Gossard, J.H. English, C.W. Tu, L. Pfeiffer and K. West, 1988, Phys. Rev. Lett. **61**, 2701.
Pinczuk, A., J.P. Valladares, D. Heiman, A.C. Gossard, J.H. English, C.W. Tu, L. Pfeiffer and K. West, 1990, Surf. Sci. **229**, 384.
Prange, R.E., 1981, Phys. Rev. B **23**, 4802.
Prange, R.E., and S.M. Girvin, eds, 1987, The Quantum Hall Effect, (Springer, New York, Berlin, Heidelberg).
Pruisken, A.M.M., 1984, Nucl. Phys. **235** [FS11], 277.
Pruisken, A.M.M., 1985, Phys. Rev. B **32**, 2636.
Pudalov, V.M., and S.G. Semenchinsky, 1984a, Solid State Commun. **51**, 19.
Pudalov, V.M., and S.G. Semenchinsky, 1984b, JETP Lett. **39**, 170.
Pudalov, V.M., and S.G. Semenchinsky, 1985, Solid State Commun. **55**, 593.
Pudalov, V.M., S.G. Semenchinsky and V.S. Edelman, 1984, JETP Lett. **39**, 576.
Pudalov, V.M., S.G. Semenchinsky and V.S. Edelman, 1985, Sov. Phys.-JETP **62**, 1079.
Puoskari, M., and A. Kallio, 1984, Phys. Rev. B **30**, 152.

Quinn, T., 1989, Metrolgia **26**, 69.
Rashba, E.I., and V.B. Timofeev, 1986, Sov. Phys. Semicond. **20**, 617.
Reatto, L., and G.V. Chester, 1967, Phys. Rev. **155**, 88.
Sachrajda, A., R. Boulet, Z. Wasilewski, P. Coleridge and F. Guillon, 1990, Sol. State Commun. **74**, 1021.
Sajoto, T., Y.W. Suen, L.W. Engel, M.B. Santos and M. Shayegan, 1990, Phys. Rev. B **41**, 8449.
Sakaki, H., K. Hirakawa, J. Yoshino, S.P. Svensson, Y. Sekiguchi, T. Hotta and S. Nishii, 1984, Surf. Sci. **142**, 306.
Schweitzer, L., B. Kramer and A. MacKinnon, 1984, J. Phys. C **17**, 4111.
Sears, V.F., 1985, Can. J. Phys. **63**, 68.
Smith III, T.P., B.B. Goldberg, M. Heiblum and P.J. Stiles, 1986, Surf. Sci. **170**, 304.
Smrcka, L., 1984, J. Phys. C **17**, L63.
Stahl, E., D. Weiss, G. Weimann, K. von Klitzing and K. Ploog, 1985, J. Phys. C **18**, L783.
Störmer, H.L., 1984, Advances in Solid State Physics, Vol. 24, ed. P. Grosse (Vieweg, Braunschweig) p. 25.
Störmer, H.L., and D.C. Tsui, 1983, Science **220**, 1241.
Störmer, H.L., Z. Schlesinger, A.M. Chang, D.C. Tsui, A.C. Gossard and W. Wiegmann, 1983a, Phys. Rev. Lett. **51**, 126.
Störmer, H.L., A. Chang, D.C. Tsui, J.C.M. Hwang, A.C. Gossard and W. Wiegmann, 1983b, Phys. Rev. Lett. **50**, 1953.
Streda, P., 1982, J. Phys. C **15**, L717.
Streda, P., and K. von Klitzing, 1984, J. Phys. C **17**, L483.
Streda, P., J. Kucera and A.H. MacDonald, 1987, Phys. Rev. Lett. **59**, 1973.
Tao, R., and F.D.M. Haldane, 1986, Phys. Rev. B **33**, 3844.
Tausendfreund, B., and K. von Klitzing, 1984, Surf. Sci. **142**, 220.
Taylor, B.N., 1989, Physics Today, **42**, 23.
Thouless, D.J., 1981, J. Phys. C **14**, 3475.
Thouless, D.J., 1982, Anderson Localization, eds Y. Nagaoka and H. Fukuyama (Springer, Berlin) p. 191.
Thouless, D.J., 1985, Phys. Rev. B **31**, 8305.
Thouless, D.J., M. Kohmoto, M.P. Nightingale and M. den Nijs, 1982, Phys. Rev. Lett. **49**, 405.
Trugman, S.A., 1983, Phys. Rev. B **27**, 7539.
Tsui, D.C., and A.C. Gossard, 1981, Appl. Phys. Lett. **38**, 550.
Tsui, D.C., and H.L. Störmer, 1986, IEEE J. Quantum Electron. QE-**22**, 1711.
Tsui, D.C., H.L. Störmer and A.C. Gossard, 1982, Phys. Rev. Lett. **48**, 1559.
Turberfield, A.J., S.R. Haynes, P.A. Wright, R.A. Ford, R.G. Clark, J.F. Ryan, J.J. Harris and C.T. Foxon, 1990, Phys. Rev. Lett. **65**, 637.
Usov, N.A., and F.R. Ulinich, 1982, Sov. Phys.-JETP **56**, 877.
van Wees, B.J., E.M.M. Willems, C.J.P.M. Harmans, C.W.J. Beenakker, H. van Houten, J.G. Williamson, C.T. Foxon and J.J. Harris, 1989, Phys. Rev. Lett. **62**, 1181.
Visscher, P.B., and L.M. Falicov, 1971, Phys. Rev. B **3**, 2541.
von Klitzing, K., 1982, Surf. Sci. **113**, 1.
von Klitzing, K., 1986, Rev. Mod. Phys. **58**, 519.
von Klitzing, K., G. Dorda and M. Pepper, 1980, Phys. Rev. Lett. **45**, 494.
Wakabayashi, J., and S. Kawaji, 1980, Surf. Sci. **98**, 299.
Wakabayashi, J., A. Fukano, S. Kawaji, K. Hirakawa, H. Sakaki, Y. Koike and T. Fukase, 1988, J. Phys. Soc. Jpn. **57**, 3678.
Washburn, S., A.B. Fowler, H. Schmid and D. Kern, 1988, Phys. Rev. Lett. **61**, 2801.
Wegner, F., 1983, Z. Phys. B **51**, 279.
Wei, H.P., A.M. Chang, D.C. Tsui and M. Razeghi, 1985a, Phys. Rev. B **32**, 7016.
Wei, H.P., D.C. Tsui and A.M.M. Pruisken, 1985b, Phys. Rev. B **33**, 1488.
Wei, H.P., A.M. Chang, D.C. Tsui, A.M.M. Pruisken and M. Razeghi, 1986, Surf. Sci. **170**, 238.
Wilczek, F., 1982, Phys. Rev. Lett. **49**, 957.
Willett, R., J.P. Eisenstein, H.L. Störmer, D.C. Tsui, A.C. Gossard and J.H. English, 1987, Phys. Rev. Lett. **59**, 1776.
Willett, R., H.L. Störmer, D.C. Tsui, A.C. Gossard and J.H. English, 1988, Phys. Rev. B **37**, 8476.

Worlock, J.M., 1990, Physics World **3**, 26.
Wu, Y.S., 1984a, Phys. Rev. Lett. **52**, 2103.
Wu, Y.S., 1984b, Phys. Rev. Lett. **53**, 111.
Yang, C.N., 1962, Rev. Mod. Phys. **34**, 694.
Yoshioka, D., 1984a, Phys. Rev. B **29**, 6833.
Yoshioka, D., 1984b, J. Phys. Soc. Jpn. **53**, 3740.
Yoshioka, D., 1986a, J. Phys. Soc. Jpn. **55**, 885.
Yoshioka, D., 1986b, J. Phys. Soc. Jpn. **55**, 3960.
Yoshioka, D., and P.A. Lee, 1983, Phys. Rev. B **27**, 4986.
Yoshioka, D., B.I. Halperin and P.A. Lee, 1983, Phys. Rev. Lett. **50**, 1219.
Yoshioka, D., A.H. MacDonald and S.M. Girvin, 1989, Phys. Rev. B **39**, 1932.
Zallen, R., and H. Scher, 1971, Phys. Rev. B **4**, 4471.
Zhang, F.C., and T. Chakraborty, 1984, Phys. Rev. B **30**, 7320.
Zhang, F.C., and T. Chakraborty, 1986, Phys. Rev. B **34**, 7076.
Zhang, F.C., and S. Das Sarma, 1986, Phys. Rev. B **33**, 2903.