General Astronomy II – PHYS1820

COMPACT OBJECTS

INTRODUCTION

Black Holes may seem mysterious, but they consist of the same ordinary matter that makes up the Sun, the Earth and everything on it. The main difference is that the matter in a black hole is squeezed into an incredibly small volume. If the Earth were to become a black hole, it would have to be compressed to the size of a marble about 1 centimetre in diameter. Newton's law of gravitation

$$F = \frac{G\, m_1\, m_2}{r^2}$$

tells us that the attractive force F between two masses m_1 and m_2 increases as the square of the distance r between the two bodies diminishes. Here on the Earth's surface we are about 6378 km from the Earth's centre. On a marble-sized Earth we would only be 0.5 cm from its centre. This huge reduction in r makes the gravitational attraction more than a billion times greater than on Earth normally.

Objects can escape from the Earth if they are shot away with speeds larger than 11 km/s. This is a tremendous velocity. But to escape a black hole, an object would need a velocity greater than the speed of light! However, according to the theory of relativity, nothing in Nature can move faster than the speed of light. In other words, not even light escapes. Black holes are truly black.

Today we have very strong evidence that there is a black hole right at the centre of our own galaxy, the Milky Way. In this exercise we will re-discover this black hole and determine its mass.

Kepler's Laws

In the early 1600s, Johannes Kepler deduced the three laws that describe the motion of the planets around the Sun. Kepler's first law describes the trajectories of the planets as ellipses, with the Sun at one focus of the ellipse. Kepler's second law relates the area A crossed by the line joining the Sun and the moving planet per unit time, such that $A/\Delta t$ = constant. Kepler's third law relates the square of the period P of the orbit in years and the cube of the semi-major axis of the elliptical orbit (which is half the distance of the longest axis of an ellipse) in units of astronomical units (AU). The orbital period of a planet is the time it takes it to make one full revolution around the Sun. It was later shown by Newton that P can be computed from:

$$P^2 = \frac{a^3}{(m_1 + m_2)}$$

where m_1 and m_2 are masses in multiples of the Sun's mass (the solar mass), and could be the mass of a star and a planet, for example. This law holds for bound orbits in the gravitational field of spherical objects, so it is also true for stars and black holes. Kepler's third law says that if you know two of the following three quantities: the period (P), the semi-major axis (a) and the total mass ($m_1 + m_2$) of the objects together, you can compute the unknown one.

The Observations

Observations of stars near the centre of the Milky Way are difficult. The many stars and dusty clouds between us and the centre obscure our view out towards the centre. Fortunately, infrared light has a longer wavelength than visible light and is much less obscured by the dusty clouds so infrared light from stars at the centre can reach us.

In successive images, taken at different times, the stars near the Milky Way centre move a bit. One star in particular, called S2, has moved a lot over the years.

The mass computation

With the positions of S2 listed in Table 1 we can determine the mass in SgrA* using Kepler's laws. Masses are mentioned in Kepler's third law, so we can use that law to find the mass of SgrA*. The law states that if you want to find the total mass $m = m_{BH} + m_S$, i.e. the mass of the black hole (m_{BH}) and the star (m_S) together, we need to know the period (P) and the semi-major axis (a) of the stellar orbit.

You will first find out the total mass and then later figure out how much belongs to the black hole and how much to the star.

You can determine the semi-major axis (a) of the stellar orbit by fitting an ellipse to the positions of S2 as listed in Table 1.

1. Plot all (x,y) positions on the graph.

2. Indicate the uncertainty on the x and y position for each point. You can do this by drawing bars with the size of the uncertainty.

3. Draw an ellipse by eye that best matches these measurements. The ellipse does not have to go through the points exactly because of the uncertainties in the positions.

4. Now measure the semi-major and semi-minor axes in arcseconds. Convert this to a length in AU using the fact that 1 arc-second corresponds to 7 098.95 AU at the Milky Way Centre. Each focus of the ellipse is offset from the center by a distance c, related to the semi-major and semi-minor axes

$$c^2 = a^2 - b^2 \qquad (1)$$

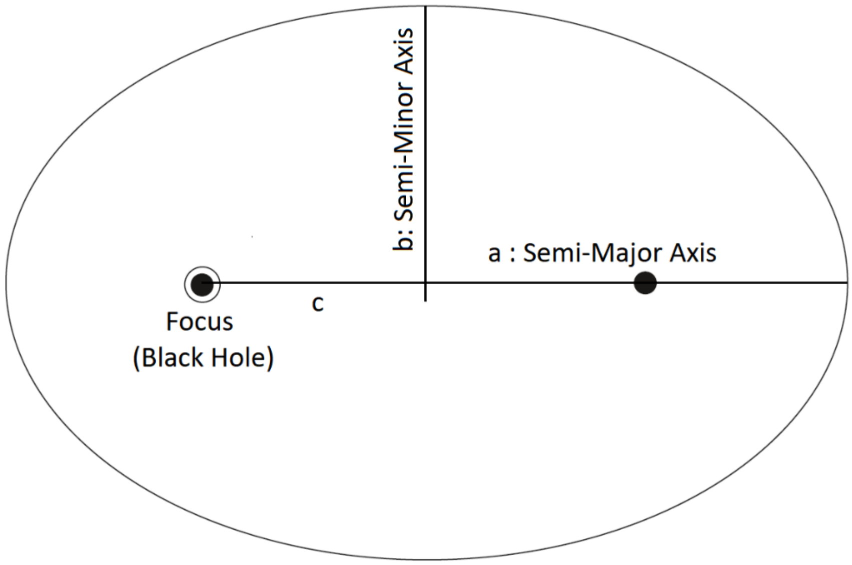

Figure 1: ***Example of an ellipse***

An example of an ellipse, showing the semi-major axis, semi-minor axis, and distance from the center of the ellipse to a focus, c.

Date (year)	x (arcsec)	y (arcsec)	Δx (arcsec)	Δy (arcsec)
1992.226	0.104	-0.166	0.003	0.004
1994.321	0.097	-0.189	0.003	0.004
1995.531	0.087	-0.192	0.002	0.003
1996.256	0.075	-0.197	0.007	0.010
1996.428	0.077	-0.193	0.002	0.003
1997.543	0.052	-0.183	0.004	0.006
1998.365	0.036	-0.167	0.001	0.002
1999.465	0.022	-0.156	0.004	0.006
2000.474	-0.000	-0.103	0.002	0.003
2000.523	-0.013	-0.113	0.003	0.004
2001.502	-0.026	-0.068	0.002	0.003
2002.252	-0.013	0.003	0.005	0.007
2002.334	-0.007	0.016	0.003	0.004
2002.408	0.009	0.023	0.003	0.005
2002.575	0.032	0.016	0.002	0.003
2002.650	0.037	0.009	0.002	0.003
2003.214	0.072	-0.024	0.001	0.002
2003.353	0.077	-0.030	0.002	0.002
2003.454	0.081	-0.036	0.002	0.002

Table 1: ***Coordinates of star S2***

Column 1: date on which the position of star S2 was measured (e.g., 2000.500 means exactly in the middle of 2000) Column 2-5: x and y positions of the star and the uncertainty in both coordinates. The units are in arc-seconds. The putative black hole is located at (0.0, 0.0)

5. Now you need to find the period (P) of the orbit of S2. In the time the star goes around the black hole once the connector – the line between the black hole and the star – maps out the area of the ellipse. The area (A_{ell}) of an ellipse is:

$$A_{ell} = \pi ab \qquad (2)$$

   Kepler's 2nd law tells you that the area traversed by the connector is proportional to the time spent in this area.

   For example, in half the period, i.e. P/2, the connector will map out half the area = $A_{ell}/2$. More generally in the time Δt it takes the star to get from position 1 to 2, the connector traverses an area:

$$\Delta A = \frac{\Delta t}{P} A_{ell} \qquad (3)$$

   To get P from this formula, you thus need to determine ΔA, Δt and A_{ell}. To do this, measure A_{ell} and the ΔA between two positions from the drawing you made in exercise 3. You can estimate areas by counting the squares on the graph paper. Now compute P from formula (3) for the two positions.

6. Now compute the total mass m of star and black hole together using Kepler's third law.

7. How much of this mass belongs to the star and how much belongs to the black hole? Stars have masses which range from 0.08 to ~120 Solar masses. The total mass computed in 6 is much larger. It is possible that this is mostly due to the black hole if the star has a negligible mass m_S compared to the mass of the black hole m_{BH} (i.e., $m_{BH} >> m_S$). However, it could also be some other object, such as a dense cluster of stars, for example.

   How many suns (N) would you need at the location of SgrA* to account for the mass? The absolute magnitude of the Sun is M_0 = +4.83. The distance to the centre of the Milky Way is about D = 8.0 kpc. The apparent magnitude of the Sun at the distance of the galactic centre is given by

$$m_0 = M_0 + 5\log(D) - 5 \qquad (4)$$

   with D in parsecs. Calculate the apparent magnitude for N suns using

$$m(N) = m_0 - 2.5\log(N) \qquad (5)$$

What would be the apparent magnitude m(N) if there were N Suns at the location of SgrA*?

Astronomers have measured almost no light coming from the Milky Way centre. These studies show that the light coming from the location of SgrA* is less than that of the surrounding stars. The conclusion is that the Milky Way centre is much too dark to allow stars to account for the measured mass. This mass must be due to a black hole.

8. Black holes do not have to be as massive as the one at the center of the Milky Way! The definition of a black hole is given as an object from which light cannot escape. The escape velocity for a spherical object with mass m and radius r can be given as

$$v_{esc} = \sqrt{\frac{2Gm}{r}} \qquad (6)$$

where $G=6.67\times10^{-11}$ m^3 kg^{-1} s^{-2}. Compute the escape velocity from the Earth using the Earth's mass $m_{Earth} = 6 \times 10^{24}$ kg and its radius R_{Earth} = 6378 km.

Now compute the escape velocity if the radius of the Earth were only 0.5 cm.

Finally, compute the escape velocity if the Earth had its usual radius but had a mass that were 2200 times that of the Sun. The mass of the Sun is about 2×10^{30} kg.

You see that the Earth is transformed into a black hole in two cases: if you compress it enormously or if you add an enormous amount of mass to it. The Sun has a radius that is a little over hundred times that of the Earth. Therefore, the second case means that you are squeezing an object with 2200 times the mass of the Sun into an object with a radius more than 100 times smaller than the Sun, in other words, also an enormous compression. The crucial property that makes a black hole a black hole is not mass or radius, but 'compactness'. This is the ratio of mass to radius, and equation (6) shows this in mathematical terms.

Worksheet:

1 arc-second corresponds to 7 098.95 AU

Semi-Major Axis (cm): ___________________

Semi-Major Axis (AU): ___________________

Semi-Minor Axis (cm): ___________________

Semi-Minor Axis (AU): ___________________

$c = \sqrt{a^2 - b^2}$ (cm): _______________

Area of ellipse: A_{ell} (cm^2 or g): ___________________

Portion of ellipse: ΔA (cm^2 or g): _________________

Time interval: Δt (years): _________________

Period: P (years): _________________

$m_1+m_2 = a^3/ P^2$

Total mass: $m_1 + m_2$ (solar masses): ______________

$m_0 = M_0 + 5\log(D) - 5$

Apparent magnitude of the Sun at the galactic center m_0: ______________

$m(N) = m_0 - 2.5\log(N)$

Apparent magnitude of N suns m(N): ___________________

Gravitational constant: $G=6.67 \times 10^{-11}$ m^3 kg^{-1} s^{-2}

Mass of the Earth: $m_{Earth} = 6 \times 10^{24}$ kg

Radius of the Earth: $R_{Earth} = 6378$ km

Mass of the Sun: $m_{Sun} = 2 \times 10^{30}$ kg

$$v_{esc} = \sqrt{\frac{2Gm}{r}}$$

Escape velocity of Earth v_{esc}: ______________

Marble Earth radius: R=0.5 cm

Escape velocity of marble Earth v_{esc}: _____________

Heavy Earth mass: $m_{HE} = 2200 m_{Sun}$

Escape velocity of heavy Earth v_{esc}: _______________